Physics III: Final (Individual)

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(Tuesday July 18, 9 AM)

First and Last Name: ________________________________

Exam Instructions:
This is an open-notebook exam, so feel free to use the notes you have transcribed throughout the summer and problem sets you have completed, but cellphones, laptops, and any notes written by someone else are prohibited. You will have **1 hour and 20 minutes** to complete this exam.

Since this is a timed exam, your solutions need not be as "organized" as are your solutions to assignments. Short calculations and succinct explanations are acceptable, and you can state (without derivation) the standard solutions to the equations of motion we derived in class. However, you should also recognize that you cannot receive partial credit for derivations/explanations you do not provide.

Problem 1: ________________

Problem 2: ________________

Problem 3: ________________

Problem 4: ________________

Total:

1. **Morse Potential (30 points)**
The interaction energy between two atoms in a diatomic molecule can be approximated by what is known as the Morse Potential:

$$U(x) = U_0 \left(1 - \beta e^{-\alpha x}\right)^2, \quad (1)$$

where $\alpha > 0$, $\beta > 0$, and $U_0 > 0$.

This potential is plotted in Fig. 1

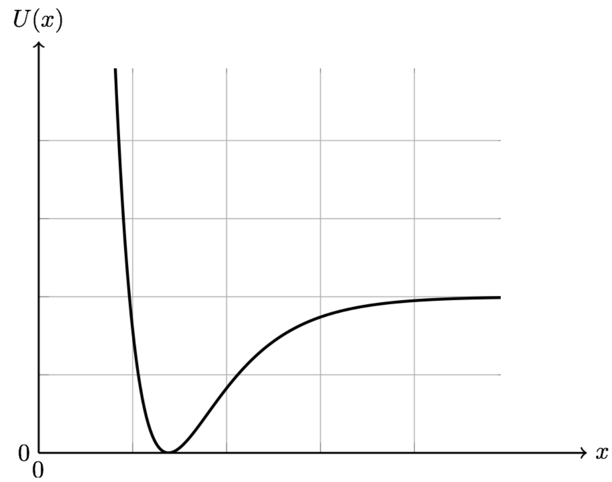

Figure 1

(a) (5 points) What are the units of U_0, β, and α?

(b) (10 points) At what value of x does $U(x)$ have a stable equilibrium?

(c) (5 points) What is the frequency of small-oscillations for a mass m, near this stable equilibrium? Given your answers in (a), show that the units of this result make sense.

(d) (10 points) Let's say the mass begins at the equilibrium position found in (b) with a velocity v_0. Assuming validity of the small-oscillation approximation, what is $x(t)$ in terms of the parameters of the system?

[Scratch work]

2. **Saturn's Rings (25 points)**
Before James Clerk Maxwell consolidated the equations of electromagnetism, he studied Saturn's rings. In this problem, we study a simple aspect of one of the dynamical motions he studied.

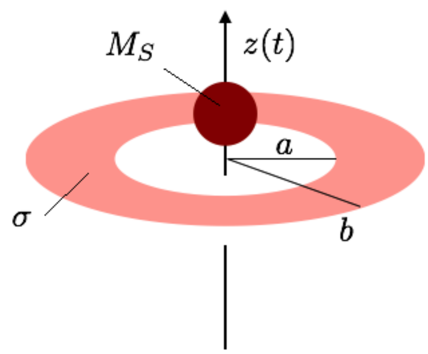

Figure 2: Depiction of Saturn-Ring system: $z(t)$ denotes the vertical position of the planet Saturn above the plane of the rings. We assume the planet is confined to move only along the z axis.

Let's say that planet saturn exhibits small oscillations about the center of the plane of its rings. Assuming the rings are defined as a disk with mass-per-area density σ, inner radius a, and outer radius b, we find that for $z(t)$ sufficiently small, the potential energy of the system[1] is

$$U(z) = U_0 + \pi G M_S \sigma \left(\frac{b-a}{ab}\right) z^2 + \mathcal{O}(z^4/a^4), \tag{2}$$

where U_0 is a constant and G is Newton's Gravitational constant.

(a) (5 points) What is the equation of motion of the mass M_S assuming it is confined to move only in the z direction? What important equation of motion is this result equivalent to?

(b) (5 points) Let's say that the mass M_S now moves through a dense cloud of particles such that it experiences a drag force

$$F_{\text{drag}} = -2\gamma M_S \, \dot{z} \tag{3}$$

for some γ. What is the equation of motion now?

(c) (10 points) We now take the motion of Saturn about the center of the plane of the rings to be an underdamped oscillator. Saturn's motion has the initial amplitude of A_0. Given the fact that Saturn begins from rest (i.e., zero velocity), determine the initial position $z(t=0)$. (*Hint: Think of the general solution to the type of equation of motion in (b)*)

(d) (5 points) Assume the motion is *very weakly* damped. In terms of the parameters in Eq.(2) and γ, determine $E(t)$ the energy of the oscillator as a function of time.

[1] If you take a college-level mechanics course, you would learn how to compute this result.

[Scratch work]

3. **Masses together (30 points)**
Say we have a system of two masses M and m which are joined by a spring of spring constant k. On its other end the mass M is attached to a spring of spring constant K_0 which is attached to a wall. We define the mass M to be at the position X and we take the mass m to be at the position x. The spring attached to the wall is in equilibrium when $X = 0$, and the spring joining the two masses is in equilibrium when $x - X = 0$.

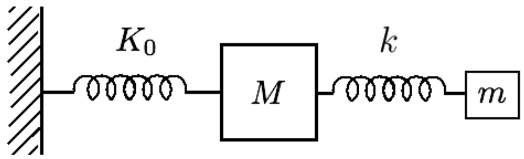

Figure 3: Two coupled masses

(a) (5 points) What are the equations of motion of the system?

(b) (10 points) What are the normal mode frequencies of this system?

(c) (5 points) We now take the spring constant $K_0 \gg k$, so that the motion of the mass m does not affect the motion of the mass M (but the converse is not true). Under this approximation, what are the two equations of motion of the system? *Hint: Assume the positions X and x are of the same order of magnitude.*

(d) (10 points) **[Given the situation in (c)]** Say the mass M begins from rest at a position $X(0) = X_0$, and the mass m begins from rest at the position $x(0) = 0$. Under the approximate equations of motion, what is the position of M and the position of m as functions of time? That is, determine $X(t)$ and $x(t)$, respectively. *(Hint: You should solve the X equation of motion first.)*

(e) (5 points) **[Given the situation in (d)]** What would the value of M have to be in order to drive the mass m at resonance?

[Scratch work]

4. **A bent string (20 points)**
A string of length L is fixed at both ends. At $t = 0$, the string is at rest and it is distorted as shown in the figure below and then released.

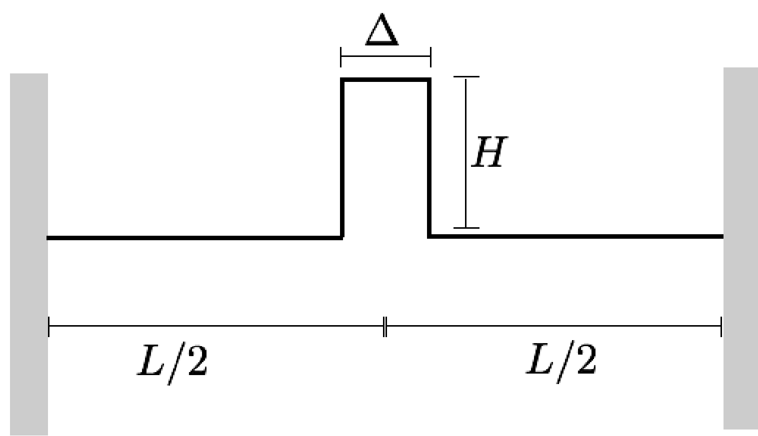

Figure 4: Bent String

(a) (5 points) Given the depiction above, what is $y(x, 0)$?

(b) (10 points) Derive an expression for the amplitude of the mth harmonic of this string. (You should be able to write your answer as a single term)

(c) (5 points) Show that for $L \gg \Delta$, the amplitude of the first few harmonics (i.e., the first few values of m) is independent of m.

[Scratch work]