Lecture 10: Linear Dynamics

In these notes, we introduce the topic of first-order linear dynamics and consider how to describe oscillatory systems outside of physics. We begin by showing that first-order differential equations in a single variable cannot exhibit oscillatory dynamics, and then we discuss two examples—a toy model and a model from population ecology—of two-variable first-order differential equations which do exhibit oscillatory dynamics.

1 Oscillations without physics?

So far in the course, we have discussed oscillating systems as defined by the fundamental equations of physics. Earlier, with Newton's 2nd Law, we derived the equations of motion of various oscillating systems and by increasing the number of degrees of freedom of the system, we were able to derive the wave equation. More recently, using Maxwell's equations we derived the wave equation for propagating electromagnetic fields.

But the world is full of oscillating systems which are not described by the equations of fundamental physics. For example, the populations of species can oscillate, the number of proteins produced in a cell can oscillate [2], and the relative proportions of reactants in a reaction can oscillate [1] and yet none of these are at all related to the equations we've been preoccupied with so far. The classic example of oscillating species populations is shown in Fig. 1. Our hope is to be able to describe (at least qualitatively) this data through a model.

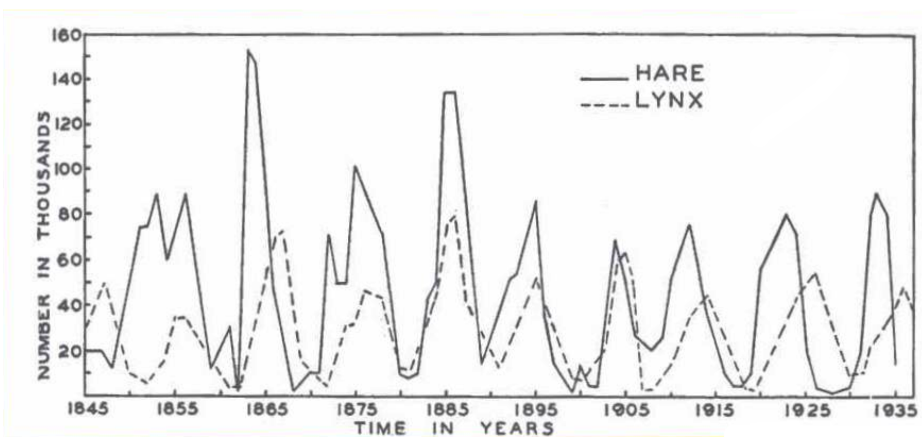

Figure 1: Data on the hare and lynx population in Hudson bay based on fur pelts collected for trading. This plot is the canonical one used for discussing the models of the type in these notes but there are reasons to be not take it too seriously. (See Reference for a short discussion of issues with data)

Finding ways to model such systems will be the focus for this final section. Phrasing this focus at a framing question, we are looking for more general ways to model oscillating systems, in particular ways which are not precisely related to physical principles.

Framing Question
How do we model oscillating systems whose dynamics are not directly defined by physical principles?

2 First-Order DE in One Variable

Systems in which variables oscillate in time are part of a wider class of systems termed **dynamical systems.** A dynamical system is simply a system in which the variables which characterize the system's degrees of freedom (e.g., positions, angles, vector fields) vary in time. In our pursuit of models of oscillatory systems, it will be useful to consider the more general class of dynamical systems first and look at oscillatory phenomena as a subset of this class.

2.1 Model of Bacterial Growth (Redux)

We will begin with the model with which we started this course: Bacterial growth. In our imaginings of our previous work in a bacteriology lab, we postulated that the growth rate of the area of a bacteria population scaled with the area itself, that is $dA/dt = kA$. Noting that the number $N(t)$ of bacteria in a colony should be proportional to the area the colony makes in the plate, the differential equation for bacterial growth could be written as

$$\frac{dN}{dt} = \lambda_0 N, \tag{1}$$

where λ_0 is a parameter with units of $1/\text{time}$. We want to know if it is possible for equations of the form shown in Eq.(1) to exhibit oscillations. Phrases more precisely, we ask

Question
Is it possible for first-order differential equations in a single variable to predict oscillatory phenomena?

Considering Eq.(1) itself, we know that this differential equation does not predict any oscillations in bacterial size: The general solution to Eq.(1) is

$$N(t) = N(0)e^{\lambda_0 t}, \tag{2}$$

which for $\lambda_0 > 0$ only predicts an exponentially growing number of bacteria[1]

What if we started with a different differential equation? Although we did not reach this conclusion when we first considered a model of bacterial growth, the solution Eq.(2) is actually unrealistic because bacteria confined to a finite container will not grow forever without bound. Instead, we should expect the growth rate to decrease the more bacteria there already are. That is if $\lambda(N)$ is the per-bacterial growth rate, instead of having $\lambda(N) = \lambda_0$ we should have something like

$$\lambda(N) = \lambda_0 \left(1 - \frac{N}{N_\infty}\right), \tag{3}$$

[1]We must also take λ_0 in Eq.(1) to be real because the growth rate dN/dt is a real and measurable quantity.

where N_∞ defines the non-zero value of N at which the bacterial population no longer grows. The differential equation for bacterial growth would therefore be

$$\frac{dN}{dt} = \lambda(N)N = \lambda_0 N\left(1 - \frac{N}{N_\infty}\right). \tag{4}$$

Does Eq.(4), exhibit oscillations in the number of bacteria? We can answer this question by finding $N(t)$. Eq.(4) can be solved exactly through the use of separation of variables (and using the identity "$a/[x(x-a)] = 1/(x-a) - 1/x$"). We quote the result:

$$N(t) = N(0)\frac{N_\infty e^{\lambda_0 t}}{N_\infty + N(0)(e^{\lambda_0 t} - 1)}. \tag{5}$$

We note that Eq.(5) does not exhibit oscillations. Rather, as $t \to \infty$ we find that $N(t) \to N_\infty$.

We could have predicted that Eq.(4) would not exhibit oscillations based on its behavior near the values of N where $dN/dt = 0$. Such values are termed **equilibrium values**[2] and we know from our previous studies of oscillatory motion, that we would only have oscillations if $N(t)$ varied as a sinusoidal function of time near its equilibria. Is this the case, for our current model? Considering the right-hand side of Eq.(4), we know that $dN/dt = 0$ when we have

$$N = 0 \qquad \text{or} \qquad N = N_\infty. \qquad \text{[Equilibrium points]} \tag{6}$$

We will consider the dynamics of our system at each of these equilibria to show why we should not have expected Eq.(4) to exhibit oscillations.

First, let's consider the dynamics near $N = 0$: We will take $N(t) = n(t)$ where $n(t) \ll N_\infty$. In this limit, Eq.(4) becomes

$$\frac{dn}{dt} \simeq \lambda_0 n, \tag{7}$$

which has the exponentially-increasing solution $n(t) \simeq n(0)e^{\lambda_0}$. Thus, near the equilibrium point $N = 0$, our bacteria system does not exhibit oscillations.

Now, let's consider the dynamics near $N = N_\infty$. We will take $N(t) = N_\infty + \delta(t)$, where $|\delta(t)| \ll N_\infty$, and $\delta(t)$ can be positive or negative. Inserting this approximation into Eq.(4) and only keeping the lowest-order non-zero terms, we find

$$\frac{d\delta}{dt} \simeq -\lambda_0 \delta, \tag{8}$$

which has the exponentially-decreasing solution $\delta(t) \simeq \delta(0)e^{-\lambda_0 t}$. Thus, near the equilibrium point $N = N_\infty$, our bacteria system does not exhibit oscillations.

2.2 Decay, Growth, or Constancy — No Oscillations

There is a pattern at work here. We could try to generalize Eq.(4) so that the growth rate $\lambda(N)$, were some other function of N, but the results would be pretty similar: Near the values of N where $dN/dt = 0$, we would find exponential-decay or exponential-increase (or even constant solution), but we would never find oscillatory dynamics. This fact is fairly important, so we'll give it it's own line

No oscillations in 1st-order single-variable DEs: First-order single variable differential equations of the form

$$\frac{dN}{dt} = F(N), \tag{9}$$

where $F(N)$ is a non-zero function of N, can yield solutions where $N(t)$ decays exponentially,

[2] This is the dynamical systems definition of equilibrium as the point where $dN/dt = 0$. This is not equilibrium in the physics sense of the word where net-force is zero. Our model of bacteria growth does not include force so the physics definition does not apply.

increases exponentially, or remains constant near its equilibria, but not solutions where $N(t)$ oscillates in time near its equilibria.

The proof of this claim is fairly simple. Say N_1 is a value of N at which $dN/dt = 0$. For example, in our second model of bacterial growth Eq.(4), N_1 would be 0 or N_∞. Let us consider the dynamics of Eq.(9) near this equilibrium value. First, we define

$$N(t) = N_1 + \delta N(t), \tag{10}$$

where $\delta N(t)$ has a magnitude much less than N_1. If N_1 defines a point where $dN/dt = 0$, then we must have $F(N_1) = 0$. Also, given Eq.(10), we can expand $F(N)$ in a Taylor series near N_1. Doing so, we find

$$F(N) = F(N_1) + F'(N_1)\delta N(t) + \mathcal{O}\left((\delta N)^2\right) = F'(N_1)\delta N(t) + \mathcal{O}\left((\delta N)^2\right). \tag{11}$$

Dropping the higher-order terms in Eq.(11) (because they are smaller corrections) and inserting Eq.(10) into Eq.(9), we then ultimately find the differential equation

$$\frac{d}{dt}\delta N(t) = F'(N_1)\delta N(t), \tag{12}$$

which has the solution

$$\delta N(t) = \delta N(0) \exp\left[t\, F'(N_1)\right], \tag{13}$$

which affirms the claim that Eq.(9) can only vary exponentially or not at all near its equilibria. If $F'(N_1) > 0$, we have exponential increase away from the equilibrium point N_1; if $F'(N_1) < 0$, we have exponential-decay toward the equilibrium point N_1; and if $F'(N_1) = 0$ the system simply remains at $\delta N(0)$ for all time.

Therefore to answer the question which framed this section: It is not possible to use equations of the form Eq.(1) (or more generally Eq.(9)) to model oscillatory dynamics. **Differential equations with one variable and, at most, a first-order derivative never exhibit oscillations.**

3 Linear First-Order DE in Multiple Variables

In the previous section, we showed that it was not possible for first-order differential equations in a single variable to lead to oscillatory dynamics. This is an important piece of information in trying to find non-physics based systems which oscillate. We already know that second-order differential equations in a single variable can oscillate. What about first-order differential equations in two or more variables?

Question
Is it possible for first-order differential equations in two (or more) variables to predict oscillatory phenomena?

We will begin with two variables. Let us say we have the system of differential equations shown below

$$\frac{dN_X}{dt} = \beta\left(N_X - N_{01}\right) + \alpha\left(N_Y - N_{02}\right), \tag{14}$$

$$\frac{dN_Y}{dt} = -\alpha\left(N_X - N_{01}\right) - \beta\left(N_Y - N_{02}\right), \tag{15}$$

where $N_X(t)$ defines the number of some quantity X, and $N_Y(t)$ defines the number of some quantity Y, both as functions of time. The system models a largely theoretical process in which the rate of increase (or decrease) of both N_X and N_Y comes in two parts. One contribution to the rate of change in N_X is proportional to $N_X - N_{01}$, so that N_X increases if $N_X > N_{01}$. A second contribution to the rate of change in N_X is proportional to $N_Y - N_{02}$, so that N_X increases if $N_Y > N_{02}$. Similarly, one contribution to the rate of change in N_Y is proportional to $-(N_X - N_{01})$ leading to decreases in N_Y, if $N_X > N_{01}$. A second contribution to the rate of change in N_Y is proportional to $(N_Y - N_{02})$ so that N_Y decreases if $N_Y > N_{01}$.

Does this system[3] exhibit oscillations in the dynamical variables N_X and N_Y? Toward answering this question we first define new variables n_X and n_Y to simplify the system of equations. With the definitions

$$n_X \equiv N_X - N_{01}, \qquad \text{and} \qquad n_Y \equiv N_Y - N_{02}, \tag{16}$$

where N_{01} and N_{02} are time independent, the system of equations becomes

$$\frac{dn_X}{dt} = \beta n_X + \alpha n_Y, \tag{17}$$

$$\frac{dn_Y}{dt} = -\alpha n_X - \beta n_Y.. \tag{18}$$

Solving the system defined by Eq.(17) and Eq.(18) would be tantamount to solving Eq.(14) and Eq.(15) because the dynamical variables are related by a constant offset shown in Eq.(16). To determine whether this system oscillates, we can find the general solutions for $n_Y(t)$ and $n_X(t)$ using methods similar to those we used to study simple harmonic oscillators and damped harmonic oscillators (i.e., guess and check methods). However, we will instead apply a more general analysis which can be extended to equations not as simple as Eq.(17) and Eq.(18).

We first represent Eq.(17) and Eq.(18) as matrix equations. We have

$$\frac{d}{dt}\mathbf{n} = \hat{\mathbf{A}}\,\mathbf{n}, \tag{19}$$

where we defined

$$\mathbf{n} = \begin{pmatrix} n_X \\ n_Y \end{pmatrix}, \qquad \hat{\mathbf{A}} = \begin{pmatrix} \beta & \alpha \\ -\alpha & -\beta \end{pmatrix}. \tag{20}$$

Our objective is to find a general solution to Eq.(19). In anticipation of any complex solutions we may find, we promote the real quantities n_X and n_Y to complex numbers $\widetilde{n}_X$ and $\widetilde{n}_Y$, respectively. The matrix $\hat{\mathbf{A}}$ remains the same, but now we have the equation

$$\frac{d}{dt}\widetilde{\mathbf{n}} = \hat{\mathbf{A}}\,\widetilde{\mathbf{n}}, \tag{21}$$

where $\widetilde{\mathbf{n}} = (\widetilde{n}_X, \widetilde{n}_Y)^T$. The $\mathbf{n}$ in Eq.(19) is related to the $\widetilde{\mathbf{n}}$ in Eq.(21) through

$$\mathbf{n} = \text{Re}[\widetilde{\mathbf{n}}]. \tag{22}$$

In order to find solutions to Eq.(21), we employ techniques of linear algebra to reduce the matrix equation to a set of single variable equations. We can achieve this by finding the eigenvalues and eigenvectors of $\hat{\mathbf{A}}$. More specifically, we assume that $\widetilde{\mathbf{n}}$ can be written as

$$\widetilde{\mathbf{n}}(t) = \widetilde{c}_1\mathbf{v}_1(t) + \widetilde{c}_2\mathbf{v}_2(t), \tag{23}$$

where $\widetilde{c}_1$ and $\widetilde{c}_2$ are arbitrary (and possibly complex) constants and $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal (and possibly complex) vectors which satisfy

$$\hat{\mathbf{A}}\mathbf{v}_1(t) = \lambda_1\mathbf{v}_1(t), \qquad \hat{\mathbf{A}}\mathbf{v}_2(t) = \lambda_2\mathbf{v}_2(t). \tag{24}$$

[3] Although the system defined by Eq.(14) and Eq.(15) is theoretical and has been chosen to investigate the possibility oscillatory dynamics in *two* variable systems, it is possible to reduce the behavior of more complicated physics systems to differential equations of this form.

Postulating the form Eq.(23) conditioned on Eq.(24) is mathematically equivalent to our previous "guess and check" method used to study coupled oscillators. The effect of guessing an exponential solution for $\widetilde{\mathbf{n}}$ essentially leads the eigenvalue-eigenvector relationships of Eq.(24). Inserting Eq.(23) into Eq.(21), using the fact that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, and employing Eq.(24), we find the differential equations

$$\frac{d}{dt}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \qquad \frac{d}{dt}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2. \tag{25}$$

These equations have the solutions $\mathbf{v}_1(t) = \mathbf{v}_1(0)e^{\lambda_1 t}$ and $\mathbf{v}_2(t) = \mathbf{v}_2(0)e^{\lambda_2 t}$, respectively. Thus we find that Eq.(23), becomes

$$\widetilde{\mathbf{n}}(t) = \widetilde{c}_1 \mathbf{v}_1(0)e^{\lambda_1 t} + \widetilde{c}_2 \mathbf{v}_2(0)e^{\lambda_2 t}, \tag{26}$$

where $\widetilde{c}_1$ and $\widetilde{c}_2$ are again arbitrary complex constants. Thus, we see that the key to finding the general solution to Eq.(21) (and, by relation, Eq.(19)), is finding the values $\lambda_{1,2}$ and $\mathbf{v}_{1,2}(0)$, that is, the eigenvalues and eigenvectors respectively of $\hat{\mathbf{A}}$ in Eq.(20).

We can find the eigenvalues and eigenvectors by the standard method. Computing the values of λ for which $\det[\hat{\mathbf{A}} - \lambda\mathbb{I}] = 0$, we have the equation

$$0 = \det\begin{pmatrix} \beta-\lambda & \alpha \\ -\alpha & -\beta-\lambda \end{pmatrix} = \lambda^2 - \beta^2 + \alpha^2 \tag{27}$$

with solutions

$$\lambda_{\pm} = \pm\sqrt{\beta^2 - \alpha^2}. \tag{28}$$

Thus we find the eigenavlues are $\lambda_1 = \sqrt{\beta^2-\alpha^2}$ and $\lambda_2 = -\sqrt{\beta^2-\alpha^2}$. Solving Eq.(24), for the corresponding (un-normalized) eigenvectors, we obtain

$$\mathbf{v}_1(0) = \begin{pmatrix} \alpha \\ \beta + \sqrt{\beta^2-\alpha^2} \end{pmatrix}, \qquad \mathbf{v}_2(0) = \begin{pmatrix} \alpha \\ \beta - \sqrt{\beta^2-\alpha^2} \end{pmatrix}. \tag{29}$$

With these results, we find that the general solution (for the complex vector $\widetilde{\mathbf{N}}$) is

$$\widetilde{\mathbf{n}}(t) = \widetilde{c}_1 \begin{pmatrix} \alpha \\ \beta + \sqrt{\beta^2-\alpha^2} \end{pmatrix} e^{t\sqrt{\beta^2-\alpha^2}} + \widetilde{c}_2 \begin{pmatrix} \alpha \\ \beta - \sqrt{\beta^2-\alpha^2} \end{pmatrix} e^{-t\sqrt{\beta^2-\alpha^2}}. \tag{30}$$

Given Eq.(30), we can determine the physical solution by using Eq.(22). Given the relationship between β and α, we have two possibilities. If $\beta > \alpha$, we can define $\Gamma \equiv \sqrt{\beta^2-\alpha^2}$, and the physical solution would be

$$\mathbf{n}(t) = c_1 \begin{pmatrix} \alpha \\ \beta+\Gamma \end{pmatrix} e^{\Gamma t} + c_2 \begin{pmatrix} \alpha \\ \beta-\Gamma \end{pmatrix} e^{-\Gamma t}, \tag{31}$$

where c_1 and c_2 are real constants. It is clear that Eq.(31) does not exhibit oscillations in the dynamical variables, for its time dependence is defined by decaying or growing exponentials. However, if $\beta < \alpha$, we can define $\Omega \equiv \sqrt{\alpha^2-\beta^2}$, and the physical solution would be

$$\mathbf{n}(t) = \begin{pmatrix} n_X(0) \\ n_Y(0) \end{pmatrix} \cos(\Omega t) + \frac{1}{\Omega}\begin{pmatrix} \beta n_X(0) + \alpha n_Y(0) \\ -\alpha n_X(0) - \beta n_Y(0) \end{pmatrix} \sin(\Omega t), \tag{32}$$

where we defined the solution in terms of $n_X(0)$ and $n_Y(0)$ for simplicity. Eq.(32) exhibits the desired behavior. Therefore, given Eq.(16) and the assumption $\alpha > \beta$, we find that N_X and N_Y oscillate about N_{01} and N_{02} with a frequency $\sqrt{\alpha^2-\beta^2}$. Thus we can answer the question which began this section in the affirmative: **It is indeed possible for a system of first-order differential equations in two variables to exhibit oscillations.**

4 Nonlinear First-Order DE in Multiple Variables

The two-variable differential equation we explored in Sec. 3 was **linear** which is to say that it only consisted of dynamical variables to the first power and no variables multiplied together. However, when studying more complex systems—like the rate equations for chemical reactions [1], mRNA and protein dynamics [2], or models of epidemics [3]—we find that the dynamical equations (even though they are often first-order in time derivatives) are **nonlinear** in the dynamical variables. This nonlinearity often leads to rich dynamical behavior which we do not have time to explore here (but check out [4] for a discussion). Instead, we will return to the situation which motivated these lecture notes, and we will show how such nonlinear equations depict oscillatory dynamics similar to the kind we have been studying all summer.

4.1 Predator Prey Model

Let's say we have an ecosystem where both hares and lynxes live. Given simple assumptions for how hares and lynxes change in number, we want to model the dynamics of the number of hares $H(t)$ and the number of lynxes $L(t)$ in this ecosystem.

We begin by considering the hares. Let's say the hares have a large food source which allows them to reproduce at a rate proportional to the number of existing hares. Thus the number of hares increases at a rate $\alpha H(t)$. However, lynxes prey on the hares. This predation occurs such that the more hares there are, the more they are eaten by lynxes, but, also, the more lynxes there are, the more they eat the hares. Thus, the number of hares decrease at a rate $\beta H(t)L(t)$. Finally, we will assume the hares die at a faster rate from being preyed upon by lynxes than they do from natural causes, so we need not include a natural death rate.

Now, let's consider the lynxes. The lynxes only reproduce if a sufficient food source is present: the more hares there are the more the lynxes will be able to reproduce. Thus the number of lynxes increase at a rate $\delta L(t)H(t)$. Finally we will assume lynxes have no natural predators in this ecosystem, and so their population only decreases from a natural death rate. Thus the number of lynxes decrease at a rate $\gamma H(t)$.

Combining these contributions to the evolution of L and H, we have the system of differential equations

$$\frac{d}{dt}H(t) = \alpha H(t) - \beta H(t)L(t) = H(t)\Big(\alpha - \beta L(t)\Big), \tag{33}$$

$$\frac{d}{dt}L(t) = \delta L(t)H(t) - \gamma L(t) = L(t)\Big(\delta H(t) - \gamma\Big). \tag{34}$$

Because Eq.(33) and Eq.(34) are nonlinear differential equations, we would have to solve them numerically in order to determine $H(t)$ and $L(t)$. However, we can make progress by considering the dynamics of the system near **steady-state**. We say our system is in steady-state if it remains the same for all time. In other words, all time-derivatives are zero. For Eq.(33) and Eq.(34), the hare and lynx population is in steady-state if

$$0 = H(t)\Big(\alpha - \beta L(t)\Big) \tag{35}$$

$$0 = L(t)\Big(\delta H(t) - \gamma\Big), \tag{36}$$

Solving Eq.(35) and Eq.(36), we find that the number of hares and the number of lynxes in the ecosystem do not change for

$$H = 0,\ L = 0 \qquad \text{or} \qquad H = \frac{\gamma}{\delta},\ L = \frac{\alpha}{\beta}. \tag{37}$$

Thus we have two steady-states in this model.

Given the fact that we can't solve Eq.(33) and Eq.(34) exactly, the question we want to answer is

Question
What are the dynamics of the hare and lynx populations when the populations are near their steady-states?

Since we have two steady-states, we have two cases to consider. We begin by considering the first case in Eq.(37). We want to approximate the dynamical equations Eq.(33) and Eq.(34) when $H(t)$ is near 0 and $L(t)$ is near 0. To that end, we define $h(t)$ and $\ell(t)$ by

$$H(t) = h(t) \qquad \text{and} \qquad L(t) = \ell(t), \tag{38}$$

where[4] $h(t) \ll 1$ and $\ell(t) \ll 1$. Thus approximating Eq.(33) and Eq.(34) near this steady-state we have

$$\frac{d}{dt}h(t) = \alpha h(t) \tag{39}$$

$$\frac{d}{dt}\ell(t) = -\gamma\ell(t) \tag{40}$$

where we dropped quadratic terms which are relatively negligible for $h(t), \ell(t) \ll 1$. Solving this system we find

$$h(t) = h(0)e^{\alpha t}, \qquad \ell(t) = \ell(0)e^{-\gamma t}, \tag{41}$$

which implies that when the hare and lynx populations are both almost dying off (i.e., both near 0), the hare population can rebound to grow exponentially while the lynxes completely die off.

Let's consider the second case in Eq.(37). We want to approximate the dynamical equations Eq.(33) and Eq.(34) when $H(t)$ is near γ/δ and $L(t)$ is near α/β. To that end, we define $h(t)$ and $\ell(t)$ by

$$H(t) = \frac{\gamma}{\delta} + h(t) \qquad \text{and} \qquad L(t) = \frac{\alpha}{\beta} + \ell(t), \tag{42}$$

where we will again take $h(t) \ll 1$ and $\ell(t) \ll 1$. Approximating Eq.(33) and Eq.(34) near this steady-state and neglecting terms of quadratic order, we find

$$\frac{d}{dt}h(t) = \frac{\beta\gamma}{\delta}\ell(t) \tag{43}$$

$$\frac{d}{dt}\ell(t) = -\frac{\alpha\delta}{\beta}h(t) \tag{44}$$

This system of differential equations can be easily solved using the methods in Sec. 3, or simply by differentiating the second equation and using the first equation. In either case, we find that the general solution is

$$h(t) = h(0)\cos\left(t\sqrt{\alpha\gamma}\right) + \frac{\beta\gamma}{\delta\sqrt{\alpha\gamma}}\ell(0)\sin(t\sqrt{\alpha\gamma}), \qquad \ell(t) = \ell(0)\cos(t\sqrt{\alpha\gamma}) - \frac{\alpha\delta}{\beta\sqrt{\alpha\gamma}}h(0)\sin\left(t\sqrt{\alpha\gamma}\right). \tag{45}$$

Thus, we see that when the hare and lynx populations are near their non-zero steady-state values, each population oscillates with a frequency $\sqrt{\alpha\gamma}$. Moreover, given the form of Eq.(45), we can infer that the two populations are out of phase by an amount which depends on α, β, γ, and δ.

Therefore, through the postulated model we have qualitatively reproduced[5] the effects shown in Fig. 1. Namely, that hare and lynx populations can oscillate and they oscillate out of phase with one another. The

[4]We will bypass the seeming nonsensical idea that there can be a number of hares and lynxes less than 1 and explore the mathematical implications of this assumption. This seeming inconsistency can be corrected by interpreting $H(t)$ and $L(t)$ as population densities rather than numbers.

[5]To obtain quantitative reproduction, we would need to estimate the values of α, β, γ, and δ.

method that we employed to study the dynamics of this nonlinear system is called **linearization**. It is very similar in spirit to the method introduced in the first assignment to perturbatively find the roots of a cubic polynomial and can thus be seen as a part of the field of **perturbation theory**.

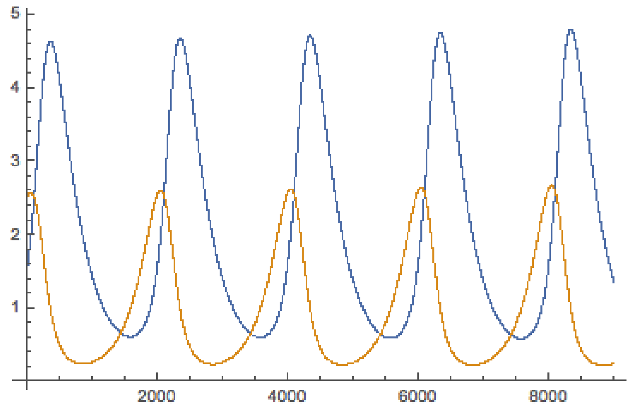

Figure 2: Plot of numerical solutions (via Euler's method) to Eq.(33) and Eq.(34). We chose $\alpha = 4$, $\beta = 2$, $\gamma = 3$, and $\delta = 3$, with $H(0) = 2.5$ and $L(0) = 1.5$. The lynx population is shown in blue, and the hare population is shown in orange. We note that the evolution of $H(t)$ and $L(t)$ is clearly periodic although the evolution is not sinusoidal. Code used to generate plot is online at http://users.physics.harvard.edu/ mwilliams/physIII_2017.html

However, we do not need to linearize our dynamical equations in order to get a sense of the dynamics in this system, and the hare and lynx populations *do not* need to be close to their steady-states to exhibit oscillations. It *is* certainly true that Eq.(45), is only valid for $H(t)$ and $L(t)$ near γ/δ and α/β, respectively, but, we can solve Eq.(33) and Eq.(34) numerically to find more general cases of oscillatory behavior. We present an example of such behavior in Fig. 2.

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