Lecture 01: Mathematical Modeling and Physics

In these notes, we define physics and discuss how the properties of physical theories suggest best practices for learning and applying them.

1 What is Physics

For the next six weeks, we will be studying physics–specifically the physics of oscillatory and wave phenomena–but before we begin, it will be useful to establish some groundwork: If we're going to be studying physics, we better know what it is and how it is different from subjects we've studied before. So before trying understand oscillatory and wave phenomena, we will explore the following questions

Framing Questions

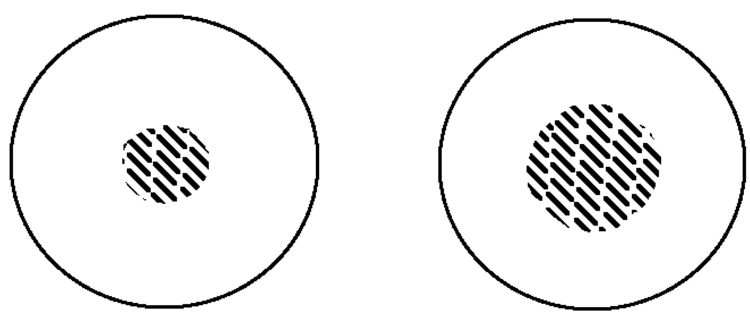
What is physics? How is it different from other disciplines? How should we study it?

To answer this question, we will begin with a non-physics example which nevertheless uses mathematics and **mathematical modeling** in very much the same way we use them in physics. This example will illustrate the basic features of how and why learning physics is different from learning biology or history.

1.1 Model of Bacterial Growth

Imagine you are a student who has acquired a solid understanding of calculus but is woefully under-read in the history and understood science of biology. You make up (or, at least, try to make up) for these deficiencies by being quite inventive and logical.

You are working in a bacteriology lab for the summer. The senior scientist at the lab tells you to to answer some question about the metabolic properties of a bacteria (which we call 'bacteria α'), but you instead become intrigued by how the bacteria are growing. Their colonies ensconced in nutrient plates seem to be expanding before your eyes.



(a) Bacterial colony at time t_0 (b) Bacterial colony at time $t_0 + 1$ hr

Figure 1: Bacterial growth in nutrient plates. The aggregate of pill shaped lines represents the bacteria colony and the large circular border represents the outline of the nutrient plates in which the bacteria grow.

In particular during the first hour, you watch the bacteria and measure the diameter of the roughly circular shape the colony makes on the nutrient plate. You take these measurements at the start and at the end of the hour, and you notice that the ratio between the final diameter $d(t_0 + 1\text{ hr})$ and the initial diameter $d(t_0)$ is ~ 1.411. You do this again in the next hour and you find that the ratio of the diameters at the end and the beginning of the hour is again 1.411[1]. This pattern in observed phenomena, suggests there should be a mathematical model associated with this system, so you ask yourself

> **Question:** How can we mathematically model the growth in area of the bacteria population?

To begin, you start consolidating your **observations**. Having mastered geometry and algebra, you recognize the measured ratio 1.411 between each subsequent diameter is approximately $\sqrt{2}$, and you also know that a circle of diameter $d(t)$ has an area $A(t) = \pi d(t)^2/4$. Thus, from your measured diameters you create the following table of your observations.

Time (hrs)	Diameter (cm)	Area (cm^2)
0	1.0	$(\pi/4)\times 1.0$
1	1.411	$(\pi/4)\times 1.990$
2	1.991	$(\pi/4)\times 3.964$
3	2.810	$(\pi/4)\times 7.892$

Table 1: Collected data for growth of bacteria

From this data you conclude that after each hour, the diameter of the bacteria population increases by a factor of $\sqrt{2}$ (i.e., $d(t + 1\text{ hr})/d(t) = \sqrt{2}$), and thus the area of the bacteria population increases by a factor of 2:

$$\frac{A(t + 1\text{ hr})}{A(t)} = \frac{\pi d(t + 1\text{ hr})^2/4}{\pi d(t)^2/4} = 2. \tag{1}$$

Now, given your knowledge of calculus, you recognize that whenever a quantity doubles in a fixed amount of time, the growth rate of the quantity must be proportional to the quantity itself. For example, when you borrow money from a bank, the money you need to pay back increases by compound interest. So that if you borrow $\$1.00$ today, then after a month you might owe $\$1.05$. For basic borrowing, the amount you owe the bank will double after a fixed amount so that as time goes on you accumulate more debt faster and faster. The basic reason for this is that the rate at which your debt increases is proportional to the amount of your current debt.

With this knowledge of how the growth rate of the area relates to the area itself, you decide to postulate a **principle** of bacterial growth. Taking $A(t)$ to be the area of the bacteria colony at a time t and $dA(t)/dt$ to be the growth rate of that area at that same time t, you write this principle mathematically as

> **Principle of Bacteria Growth (Mathematical Formulation):** If a colony of bacteria α is in a culture dish with sufficient nutrients, then the area A of the colony evolves in time according to
>
> $$\frac{dA(t)}{dt} = kA(t), \tag{2}$$
>
> for some k of units $1/\text{hr}$.

Eq.(2) is a good starting point for a mathematical model of bacterial growth. However, it contains an unknown quantity k. Fortunately, you surmise that it should be possible to determine the value of k by making proper use of your observations in Table 1. Namely, given the data in 1 you want to know what the theory represented by Eq.(2) predicts for how long it takes the area of the bacterial colony to double; this prediction will in turn constrain the value of k.

[1] If we were being careful about this, we would include **error bars** with this measurement, but we will forego them for this example.

Again, given your calculus knowledge, you start by solving the differential equation Eq.(2). To do so, you follow the basic algorithm:

$$\begin{aligned}
\frac{dA(t)}{dt} &= kA(t) \\
\frac{1}{A(t)}\frac{dA(t)}{dt} &= k \quad \text{[Divide by } A(t)\text{]} \\
\int_{t_0}^{t_f} \frac{1}{A(t)}\frac{dA(t)}{dt}dt &= k\int_0^{t_f} dt \quad \text{[Integrate both sides from } t=0 \text{ to } t=t_f\text{]} \\
\int_{A_0}^{A(t_f)} \frac{1}{A}dA &= k\int_{t_0}^{t_f} dt \quad \text{[Change variables in left integral]} \\
\ln\frac{A(t_f)}{A_0} &= kt_f \quad \text{[Compute integrals on both sides].} \qquad (3)
\end{aligned}$$

Taking the exponential of both sides of Eq.(3), you thus surmise that for a general time t the area of the bacterial population grows as

$$A(t) = A_0 e^{kt}, \qquad (4)$$

where A_0 is the area of the colony at the chosen initial time $t = 0$. If the area of the colony doubles after a time t_d (which for this case is 1 hr), you realize that by Eq.(4) $A(t_d)$ must equal $2A_0$. Thus, you find

$$2 = \frac{2A_0}{A_0} = \frac{A(t_d)}{A_0} = \frac{A_0 e^{kt_d}}{A_0} = e^{kt_d}. \qquad (5)$$

Taking the far LHS and the far RHS and solving for k, you obtain

$$k = \frac{\ln 2}{t_d}. \qquad (6)$$

With your previous observations, consolidated into Eq.(1), you know that the doubling time is $t_d = 1$ hr. You thus claim k is given by

$$k = \ln 2\ \text{hr}^{-1}. \qquad (7)$$

With this result, you have answered your original question. The area of the bacteria population grows according to Eq.(4) where k is given by Eq.(7) and A_0 depends on our choice of an initial time.

But you're not done yet. Now, you now want to see what else you can do with this principle. Knowing that the bacteria are in a finite circular plate of diameter 15 cm, you realize there is a constraint in this setup on how much the bacteria can grow according to Eq.(2). You decide to test this by predicting how long it would take the bacteria (which currently comprises an area with diameter of about 3 cm) to reach the limits of the culture dish. Given Eq.(4) and Eq.(7), you **predict** it should take a time

$$t_f = \frac{1}{k}\ln\frac{A_f}{A_0} \approx 4.6\ \text{hr}, \qquad (8)$$

where $A_f/A_0 = (15/3)^2 = 25$, to reach the limits of the dish. You look at your watch to check the time and finally decide to leave the lab and go on a very long lunch. When you return...

1.2 Structure of Physics

Well you (reader) get the idea. This example was meant to illustrate the dual processes of induction and deduction and how they allow us to mathematically model and make predictions about the world. Moreover, although this example doesn't at all deal with physics, it is nevertheless an example of mathematical modeling and therefore provides a conceptually simple framework by which to develop a structural understanding of physics.

- **Induction and Deduction: Induction** refers to the process of developing or postulating general laws based on specific observations. **Deduction** refers to the process of deriving specific predictions from general laws.

  Although induction and deduction do not characterize all of science, the two concepts largely describe how physicists (and quantitative scientists in general) formulate theories.

  For example, in the previous section, a theory of bacterial growth was formulated by first observing the growth of bacteria, extrapolating a principle from the properties of this growth (induction), and then using the principle to make a testable prediction (deduction). We depict the cyclical nature of this process below.

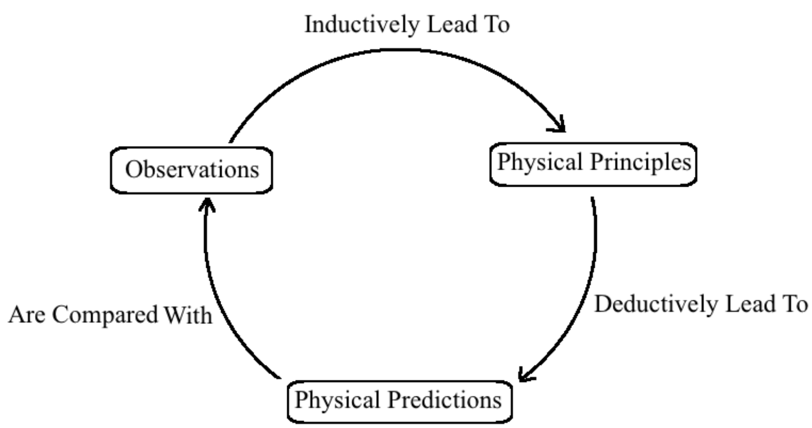


Figure 2: The cyclical relationship between the observations which motivate the development of principles which are, in turn, used to obtain predictions which are then checked against more observations.

  It is important to take note of this structure because it largely characterizes how physicists in all disciplines formulate theories. For example, Newton's laws are principles (gleaned from observations of physical phenomena) which produce predictions which can be compared with other observations. We should mention that this process in practice is rarely ever this clean, and the physicist often jumps between stages as he works through trial-and-error to find the proper principles or predictions by which to model a phenomena.

  Also, sometimes physicists don't directly use observations to develop physical principles but rely on intuition and abstractions. Einstein's formulations of Special and General Relativity began with such an intuition [1].

- **Principles and Predictions:** This example was also meant to illustrate the difference (with regard to a hierarchy of importance) between principles and predictions. Principles are taken as assumptions and are often used as the starting points of a theory. From these starting points one extends the theory in various directions to obtain predictions (see Fig. 3 below). In this way we consider principles as more fundamental than predictions. However, understanding a theory often amounts to understanding both the principles and predictions, in addition to the ways they are connected.

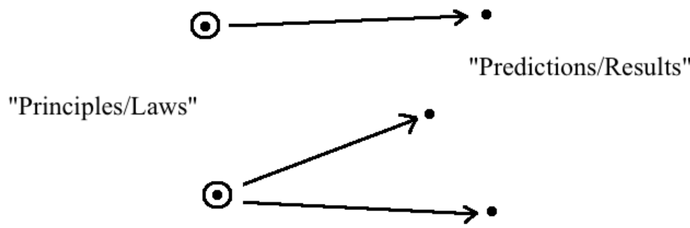


Figure 3: Predictions in a theory extend from and are less "fundamental" than the principles. This does not mean predictions are less important; only that they are not the deductive starting point of the theory. In this diagram the arrows stand for a mathematical derivation.

We can illustrate the relationship between predictions and principles with a non-physics related example. Consider the following valid logical argument:

1. A is true.
2. If A is true, then B is true.
3. B is true

In this argument, statement 1 serves the role of a **premise** which means it is not derived from any other statement; to consider the validity of this argument, we simply take the premise as true. Similarly, in physics, physical principles and laws are taken as true without any prior logical[2] justification.

Statement 3 is the conclusion of this logical argument since it is deduced from the premise. The **conclusion** of a logical argument is analogous to a physical prediction or derived result in physics because these physical predictions are deduced from physical principles.

What role does statement 2 play? It acts as a logical connection between statements 1 and 3 and thus requires us to accept the truth of statement 3 if we accept the truth of statement 1. In physics, mathematics serves the role of statement 2.

- **Derivations and Mathematics:** Some people state that mathematics is the language of physics to emphasize that, in a way very similar to how written language can be used to express qualitative ideas, mathematics can be used to express ideas about change, geometry, and structure. However, in physics, mathematics is more than just a symbolic method for expressing physical ideas; it also serves to embed these idea in a sophisticated logical framework which then allows these ideas to be connected to other ideas.

  We see this most clearly in the way physical principles, expressed mathematically, allow us to derive predictions for a mathematical theory. For example, when we we modeled bacterial growth, we used calculus to not only *express* our starting principle as the differential equation Eq.(2), but to also *derive* the prediction Eq.(8).

  More generally the mathematics we use to make predictions in a theory vary by subject. To illustrate this relationship, in the table below we list a few physical theories along with some of their associated physical predictions and the mathematical frameworks used to derive them. We include our model of bacteria growth (which is NOT a physical theory) as a comparison.

[2]By "logical" we mean the philosophical definition of "logic" rather than the colloquial meaning which is synonymous with "rational".

Table 2: Physical Theories: Principles and the Mathematics used to obtain Predictions

	Principles	Mathematics	Example Result/Prediction
Bacterial Growth	Principle of Bacterial Growth	Calculus	Time for Growth
Classical Mechanics	Newton's Laws; Newton's Law of Gravitation	Calculus; Linear Algebra	Period of Pendulum
Electromagnetism	Maxwell's Equations; Lorentz Force Law	Vector Calculus; Partial Differential Equations	Electric field of General Conductor

Table 3: *Our model of bacteria growth is not an actual theory because it does not follow from a physical principle.

2 Considering and Learning Physics

(a) (b)

Figure 4: Two types of understanding: (a) Taking ideas and equations as distinct and learning them as such. Leads to memorization and an inability to extend knowledge. (b) Understanding connections between ideas and equations. Allows for less memorization and develops skills to extend knowledge.

This discussion of the structure of physics is important because it motivates a particular way of studying and engaging with the subject. In particular, it should discourage an unfortunately typical way of learning the subject and encourage a way which is more natural and, in the long run, more useful.

In some physics classes, the main results which define a subject are sometimes employed in a disconnected manner which suggests that these results were found and hence should be applied independently of one another. A standard example is "The Big Four" equations of kinematics which are often presented as distinct. However, these kinematic equation can all naturally be subsumed into the equation for constant acceleration:

$$\left.\begin{aligned} x(t) &= x_0 + v_0 t + \tfrac{1}{2}at^2 \\ v(t) &= v_0 + at \\ v_f^2 &= v_0^2 + 2a(x_f - x_0) \\ x(t) &= x_0 + \tfrac{1}{2}(v(t) + v_0) \end{aligned}\right\} \quad \rightarrow \quad \frac{d^2}{dt^2}x(t) = a \tag{9}$$

This is general in physics; all results in a physical theory (outside the physical principles and assumptions) are derived from and hence connected to other results. Thus in learning physics we should not memorize the results independent of one another, but should rather understand how they are connected. We depict this perspective in Fig. 4

Fig. 4 is meant to suggest that what you learn in a subject becomes more useful when you understand how the topics and ideas which define the subject are connected. One benefit of learning this way is that if you forget something (e.g., imagine if a node in Fig. 4 was erased), then since you understand how results in the subject are connected, you can rederive what you have forgotten from all that you still know.

To put it plainly: **only memorizing equations is an inefficient way to learn a subject which has as much manifest logical structure as physics does. Instead, efficiently learning physics entails learning the structure of the subject and how theorems and physical results are derived and are related to one another.**

Therefore, understanding in physics is not merely represented by what equations or ideas you know, but more truly in what you know about the connections between these equations and ideas. It is only by understanding these existing connections that you will ever have the knowledge and skills to move beyond them toward a comprehension of something you have never before seen.

3 What is Physics? Part II

Our model of bacterial growth, although being a solid piece of mathematical modeling, cannot be considered physics because the model is not grounded in any **physical principles**[3]. Examples of physical principles you've heard of before are

- Laws of Thermodynamics
- Schrödinger Equation
- Newton's Second Law

Thus given that we get physics when we combine physical principles with mathematical modeling, we can now answer the question posed at the very beginning of these notes:

> **Physics (definition):**
> Physics is a scientific discipline which uses mathematically formulated physical principles to model, explain, and predict phenomena in the inanimate world.

For example, the various foundational subject in physics can answer questions such as

- Why do hot objects emit light?
- Why are certain elements non-reactive?
- Why is the period of a pendulum independent of its mass?

In this class we will be concerned with questions of the last kind, namely questions which deal oscillatory phenomena. As in the non-physics example, presented in these notes we will use observable situations to motivate our exploration of each new topic we study. As always we will emphasize how the topics are connected to the principles of physics and also attempt to understand some basic physical results.

References

[1] A. Einstein, "On the method of theoretical physics," *Philosophy of science*, vol. 1, no. 2, pp. 163–169, 1934.

[3] We introduced the "principle of bacterial growth" is not a physical principle. It was introduced to to illustrate how principles are used in mathematical modeling