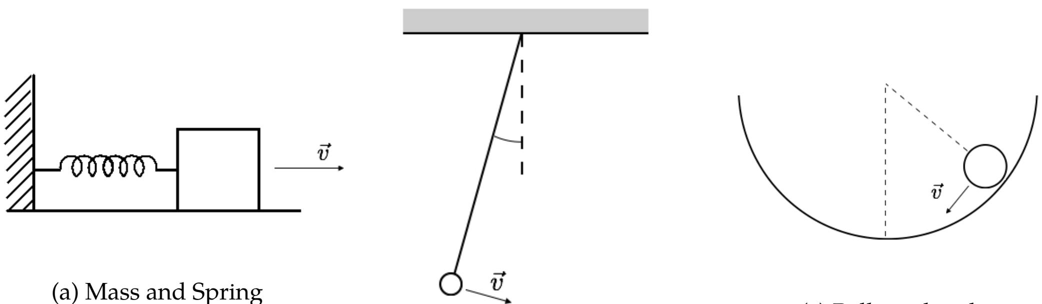
Lectures 02 and 03: Simple Harmonic Oscillator, Classical Pendulum, and General Oscillations

In these notes, we introduce simple harmonic oscillator motions, its defining equation of motion, and the corresponding general solutions. We discuss how the equation of motion of the pendulum approximates the simple harmonic oscillator equation of motion in the small angle approximation.

1 Simple Harmonic Oscillator

Consider the three scenarios depicted below:



(a) Mass and Spring

(b) Pendulum

(c) Ball in a bowl

Figure 1: Three different systems which exhibit simple harmonic motion. The velocity vector $\vec{v}$ is identified in each to define the direction of motion.

In Fig. 1a we have a mass attached to a spring and moving back and forth horizontally. In Fig. 1b we have a heavy ball attached to a much less massive string and swinging about a pivot.In Fig. 1c we have a ball rolling around near the bottom of a hemispherical bowl. What do all of these phenomena have in common? Under ideal conditions[1] they are all examples of **simple harmonic motion** which is characterized by having an acceleration which is proportional to but in the opposite direction of the position. For the first part of this course we will be attempting to model this motion under various conditions, so our framing question is

Framing Question

How can we use physical principles to understand and model simple harmonic motion?

1.1 Kinematics Review:

Before we discuss the simple harmonic oscillator, let us review some basic concepts in classical mechanics. When attempting to describe the trajectory of a particle moving in multiple dimensions, we define the

[1]Conditions where friction is negligible and the objects are near equilibrium (defined later).

particle's **position** by a $\vec{r}(t)$, a function of time with units of distance. In three dimensions $\vec{r}(t)$ has three components and can be written as

$$\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}, \tag{1}$$

where $x(t)$, $y(t)$, and $z(t)$ define motion in the x, y, and z direction respectively, and $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors associated with the relevant direction. To determine the rate at which the position is changing with time, we compute the velocity $\vec{v}(t)$ which is defined as the derivative of the position:

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \frac{d}{dt}\vec{r}(t) \equiv \dot{\vec{r}}(t). \tag{2}$$

In the final equivalence in Eq.(2), we have introduced **dot notation** to simplify the form of the derivative operator. From now on, we will use a dot above a variable to signify the time derivative of that variable; correspondingly, nth-order time derivatives are identified by n dots.

The limit in Eq.(2) implies that the velocity $\vec{v}(t)$ is the instantaneous rate of change of the position $x(t)$. Similarly, to determine the rate at which the velocity is changing with time, we compute the acceleration which is defined as the derivative of the velocity:

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d}{dt}\vec{v}(t) = \dot{\vec{v}}(t). \tag{3}$$

Because the acceleration is the derivative of the velocity and the velocity is the derivative of the position, the acceleration is the second derivative of the position:

$$\vec{a}(t)(t) = \frac{d}{dt}\frac{d}{dt}\vec{x}(t) = \frac{d^2}{dt^2}\vec{x}(t) = \ddot{\vec{x}}(t). \tag{4}$$

We could also write these relationships as integration formulas. For a function $f(x)$ such that $f(x) = dF(x)/dx$, the Fundamental Theory of Calculus states

$$\int_a^b dx\, f(x) = F(b) - F(a), \tag{5}$$

where b and a are points in the x domain. We note that in Eq.(5), x is a **dummy integration** variable and hence does not change the value of the integral on the left-hand side. We could, for example, replace x with another variable u or y and we would get the same result. Given, Eq.(5) we can rewrite Eq.(2), Eq.(3), and Eq.(4) as integration formulas. Doing so, we find, respectively,

$$\vec{x}(t) = \vec{x}_0 + \int_{t_0}^{t} dt'\,\vec{v}(t'), \quad \vec{v}(t) = \vec{v}_0 + \int_{t_0}^{t} dt'\,\vec{a}(t'), \quad \text{and} \quad \vec{x}(t) = \vec{x}_0 + \vec{v}_0 t + \int_{t_0}^{t} dt' \int_{t_0}^{t''} dt''\,\vec{a}(t''). \tag{6}$$

We note that in Eq.(6) t' and t'' serve as dummy variables which are used to parameterize the integration but which do not affect the result of the integration. To obtain the last equation in Eq.(6), we plugged the second equation into the first equation.

The above formulas are relevant when we study Newtonian Mechanics for they allow us to characterize the various properties of motion in our system of study. In particular Newton's 2nd Law states that the net force (i.e., the sum of all forces) $\vec{F}_{\text{net}}$ acting on a particle of mass m is related to the acceleration of the particle through

$$m\ddot{\vec{r}}(t) = \vec{F}_{\text{net}}. \tag{7}$$

Net force F_{net}—like position, velocity, and acceleration—is a **vector quantity** meaning it has both magnitude and direction; for one-dimension this means both its absolute value and its sign are physically important.

Eq.(7) defines the **equation of motion** of our mechanical system. The equation of motion of a system is always given by a differential equation which defines how the position changes in time. When we solve the differential equation and find $x(t)$ (or the value of whatever analogous kinematic variable defines our system) we can then completely characterize all kinematic aspects of the system.

We will be spending much of this class solving equations of motion. To provide practice in this direction, we solve one you might have seen before. Say we have a particle of mass m falling in a gravitational field. The force exerted on the particle is

$$\vec{F} = -mg\,\hat{y}, \tag{8}$$

where g is the gravitational acceleration constant and we have taken $+\hat{y}$ to define the positive vertical direction. Taking $\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y}$ to define the position of the particle and using Eq.(7), we find the equation of motion

$$\begin{aligned} m\ddot{\vec{r}}(t) &= \vec{F} \\ m\frac{d^2}{dt^2}\left(x(t)\,\hat{x} + y(t)\,\hat{y}\right) &= \vec{F} \\ m\left(\ddot{x}(t)\,\hat{x} + \ddot{y}(t)\,\hat{y}\right) &= -mg\,\hat{y} \\ m\ddot{x}(t)\,\hat{x} + m\ddot{y}(t)\,\hat{y} &= -mg\,\hat{y} \end{aligned} \tag{9}$$

Because the directions $\hat{x}$ and $\hat{y}$ are independent, we can separate Eq.(9) into two equations, one for each direction:

$$m\frac{d^2}{dt^2}x(t) = 0, \qquad m\frac{d^2}{dt^2}y(t) = -mg. \tag{10}$$

To solve these equations, we use the formulas given in Eq.(6). We are only interested in a single direction for each equation in Eq.(10), so we can neglect the vector notation in Eq.(6). Using these integration formulas, we find the solutions

$$v_x(t) = v_{0x}, \qquad v_y(t) = v_{0y} - gt, \tag{11}$$

$$x(t) = x_0 + v_{0x}t, \qquad y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2, \tag{12}$$

where v_{0x} and v_{0y} are the initial velocity in the x and y direction respectively. If we parameterize these initial velocities by the angle θ (i.e, the angle at which the mass m is launched into the air), then we can define the these velocities in terms of the speed v_0 (defined as the magnitude of the velocity vector) and the angle θ:

$$v_{0x} = v_0\cos\theta, \quad v_{0y} = v_0\sin\theta. \tag{13}$$

Using Eq.(13) and Eq.(12), we can derive that the trajectory of the particle giving the vertical position y as a function of horizontal position x is

$$y(x) = y_0 + (x - x_0)\tan\theta - \frac{g(x-x_0)^2}{2v_0^2\cos^2\theta}. \tag{14}$$

1.2 SHO equation of motion

Having reviewed some basic results in kinematics and equations of motion, we are now ready to consider the titular motion of these notes. We will consider the first scenario in Fig. 1 redrawn in Fig. 2. We have a mass m attached to a spring which is itself attached to a wall. The mass, initially at rest and at the equilibrium position $x(t) = x_{\text{eq}}$, is pulled to a position $x(t) = x$. For the simplest type of spring, if $x - x_{\text{eq}}$ is sufficiently

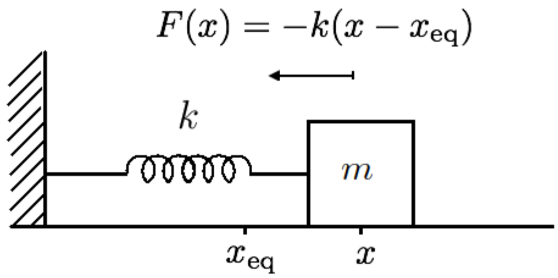


Figure 2: Mass-spring system. By Hooke's law, the spring exerts a force on the mass which always points toward the equilibrium position and is proportional to the displacement from that position.

small[2], the force exerted by the spring on the mass is

$$F = -k(x - x_{\text{eq}}), \tag{15}$$

where k is a constant with units of N/m. Eq.(15) is **Hooke's law**. It states that, for so-called Hookean springs, the force exerted by the spring on a mass is proportional to the mass's displacement from the equilibrium position, and this force always points toward the equilibrium position. In order to precisely describe the motion determined by Eq.(15), we need to solve Newton's 2nd law given this force. For simplicity we will take $x_{\text{eq}} = 0$; its actual value depends on the setup of our system, but doesn't change the resulting dynamics. For our one-dimensional system, in which the mass is only acted upon by the spring, Newton's second law gives us

$$m\ddot{x}(t) = F_{\text{net}} = -kx(t), \tag{16}$$

where we explicitly wrote the time dependence of the position for clarity. Adding $kx(t)$ to both sides of Eq.(16), and dividing both sides by m, we then have the equation

$$\ddot{x}(t) + \frac{k}{m}x(t) = 0. \tag{17}$$

We are almost at a general equation which can model the situations shown in Fig. 1, but before we get there we need to write Eq.(17) somewhat more abstractly. The constants k and m are specific to the spring-mass system, but what their ratio represents is not. Given the units of k and m separately, we find that the units of the quantity k/m are

$$\left[\frac{k}{m}\right] = [k] \times [m]^{-1} = \frac{\text{N}}{\text{m}} \times \frac{1}{\text{kg}} = \frac{\text{kg} \cdot \text{m}}{\text{s}^2}\frac{1}{\text{m}} \times \frac{1}{\text{kg}} = \frac{1}{\text{s}^2}. \tag{18}$$

Given that k/m has units of $1/\text{s}^2$ and frequency has units of $1/s$, we term $\sqrt{k/m}$ the **angular frequency**[3] of our system, and define

$$\omega_0 \equiv \sqrt{\frac{k}{m}}, \tag{19}$$

where ω_0 (pronounced "oh-meh-ga") stands in for the angular frequency. Thus Eq.(17) becomes

$$\ddot{x}(t) + \omega_0^2 x(t) = 0. \tag{20}$$

[2]What we mean by sufficiently small depends on unstated parameters which define the spring force.
[3]We will later find that $\sqrt{k/m}$ has the units radians/sec, but radians are always taken to be dimensionless.

Eq.(20) is termed the **simple harmonic oscillator (SHO) equation of motion**. It is the most basic equation of the collection of equations involving mechanical oscillations. Moreover, the system Eq.(20) describes (i.e., a system of small oscillations about an equilibrium position) is ubiquitous throughout physics; it appears in classical mechanics, quantum mechanics, thermodynamics, and electrodynamics.

1.3 Solving equation of motion

Eq.(20) defines the dynamics of our system; it tells us how the position changes in time given the spring constant k and mass m. Now, we want to determine the kinematics of the system, that is we want to know what the position is as a function of time. Thus, we need to solve Eq.(20), and we will know we have succeeded in doing so when we find the *most* general $x(t)$ which satisfies Eq.(20).

The standard method of solving equations of the form Eq.(20) is actually quite simple. Such equations are called **homogeneous linear differential equations with constant coefficients**. Here's the name break down (along with examples of equations which are not homogeneous, are not linear, or do not constant-coefficients) :

- **Homogeneous:** refers to the fact that *all* terms in the differential equation include the function $x(t)$.
  Inhomogeneous equation example: $\ddot{x}(t) + \omega_0^2 x(t) = F_0 \cos(\omega t)$.
- **Linear:** refers to the fact that all terms have $x(t)$ raised to the first power.
  Nonlinear equation: $\ddot{x}(t) + \omega_0^2 x(t) - \lambda x(t)^2 = 0$.
- **Constant Coefficients:** refers to the fact that all coefficients in the differential equation are independent of t and x.
  Time-dependent coefficients example: $\ddot{x}(t) + \omega_0^2 \sin(\omega t) x(t) = 0$.

Homogeneous linear differential equations with constant coefficients always have solutions which are exponential functions because exponential functions have the unique property that they are their own derivative. Given this property of exponentials, the way we solve Eq.(20) is to guess a general exponential function with arbitrary parameters and then choose the parameters such that they satisfy the equation. Because Eq.(20) defines oscillatory motion that is equivalent to the simplest type of periodic motion, we expect that the $x(t)$ that solves Eq.(20) should be composed of sine functions and cosine functions. We will find this is indeed the case.

Our starting guess will be

$$x(t) \stackrel{?}{=} Ae^{\alpha t} \qquad \text{[Guessed solution]}, \tag{21}$$

where A and α (pronounced "al-fah") are arbitrary constants that we may need to determine. Inserting, Eq.(21) into Eq.(20), we find

$$\begin{aligned} 0 &= \ddot{x}(t) + \omega_0^2 x(t) \\ &= \frac{d^2}{dt^2}\left[Ae^{\alpha t}\right] + \omega_0^2 A e^{\alpha t} \\ &= \alpha^2 A e^{\alpha t} + \omega_0^2 A e^{\alpha t} \\ &= (\alpha^2 + \omega_0^2) A e^{\alpha t}. \end{aligned} \tag{22}$$

Because $e^{\alpha t}$ has the property that it can never be zero for finite t, we can divide out $Ae^{\alpha t}$ in the final equality. Thus in order for Eq.(21) to solve the SHO equation of motion we need to solve

$$\alpha^2 + \omega_0^2 = 0, \tag{23}$$

for α. Doing so, we find that α can take on two values: We can have $\alpha = +i\omega_0$, or $\alpha = -i\omega_0$, where $i \equiv \sqrt{-1}$ is the imaginary unit. Both options of α are valid, and both yield a Eq.(21) which solves Eq.(20). Therefore, we have the two, purely exponential solutions

$$x(t) = A_+ e^{i\omega_0 t} \quad \text{or} \quad x(t) = A_- e^{-i\omega_0 t}. \tag{24}$$

where A_+ and A_- are two different coefficients corresponding to the two different possible values of α. We do not choose A_+ and A_- to be the same because $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ are different functions and can thus be seen as two different guesses of the form Eq.(21).

Both functions in Eq.(24) satisfy Eq.(20). What then is the most general solution? It turns out the most general form is a sum of the two solutions in Eq.(24), that is

$$x(t) \stackrel{?}{=} A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}, \tag{25}$$

where A_+ and A_- are two undetermined and independent constants. We can check Eq.(25) by inserting it into Eq.(20) and ensuring it correctly satisfies the differential equation.

But we include a question mark in Eq.(25) because we are not quite finished. The quantities $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ are complex and therefore contain imaginary numbers, but $x(t)$ is a real quantity which we can measure. Thus, the general solution Eq.(25) is not a physical solution until we take its real part. In order to do so, we need to review some facts about complex exponentials.

1.4 Aside/Review: Complex exponentials

In order to find the most general *physical* solution of Eq.(20), we need to take the real part of Eq.(25). Before we do so we need to establish an important identity. Let's say we have the following differential equation

$$\frac{d}{d\theta} f(\theta) = i f(\theta), \tag{26}$$

with the condition $f(\theta = 0) = 1$. What is the solution to this equation? Guessing a solution, we find that

$$f_1(\theta) = e^{i\theta} \tag{27}$$

is a valid solution because $f(0) = e^0 = 1$ and

$$\frac{d}{d\theta} f_1(\theta) = i e^{i\theta} = i f_1(\theta). \tag{28}$$

But another solution that works is

$$f_2(\theta) = \cos\theta + i \sin\theta. \tag{29}$$

This solution also satisfies the condition $f(\theta = 0) = 1$ and it also gives us

$$\frac{d}{d\theta} f_2(\theta) = -\sin\theta + i\cos\theta = i(i\sin\theta + \cos\theta) = i f_2(\theta). \tag{30}$$

Since both Eq.(27) and Eq.(29) are two solutions to the first-order equation Eq.(26) and both satisfy the condition $f(0) = 1$, they must represent the same solution. Thus, we have the identity[4]

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{31}$$

Eq.(31) is termed **Euler's** (pronounced "Oy-ler"s) **formula**. Eq.(31) is a fundamental result of complex algebra and is useful in proving many trigonometric identities. It is given much fanfare in mathematics literature, because when we plug $\theta = \pi$ into Eq.(31), we find **Euler's identity**

$$e^{i\pi} + 1 = 0. \tag{32}$$

[4]The other standard way to prove this identity is to use Taylor series.

This equation is seen as beautiful by many math-appreciators because it contains three mathematical constants from three areas of mathematics: i arises from algebra; e extends from calculus; π was first defined in geometry.

With regard to proving trigonometric identities, Eq.(31) provides a simple way, for example, of establishing sum of angle formulas. Take the identity $\cos\theta/2 = \sqrt{(1+\cos\theta)/2}$. This identity can be easily proven using complex exponentials:

$$\begin{aligned}\left(e^{i\theta/2}\right)^2 &= (\cos(\theta/2) + i\sin(\theta/2))^2 \\ e^{i\theta} &= \cos^2(\theta/2) - \sin^2(\theta/2) + i\left(2\cos(\theta/2)\sin(\theta/2)\right) \\ \cos\theta + i\sin\theta &= 2\cos^2(\theta/2) - 1 + i\left(2\cos(\theta/2)\sin(\theta/2)\right)\end{aligned} \tag{33}$$

Taking the real part of both sides in the last line in Eq.(33), we find

$$\cos\theta = 2\cos^2\theta/2 - 1, \tag{34}$$

which yields one of the half-angle formulas.

Eq.(31) is also useful in writing complex numbers in a very compact form. In general, the complex number $a + bi$ can be written as

$$a + bi = re^{i\theta}, \tag{35}$$

where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$, with the ambiguity in θ resolved by inspecting the quadrant in which $a + bi$ lies.

1.5 Solving equation of motion-part 2

With Eq.(31), we can now find the physical part of the solution Eq.(25). This physical part is simply the real part of Eq.(25). To find this real part we use the identity (which you should check)

$$\text{Re}\left[(a_1 + ib_1)(a_2 + ib_2)\right] = a_1a_2 - b_1b_2. \tag{36}$$

Taking A_+ and A_- to be complex numbers defined as

$$A_+ \equiv B_+ + iC_+ \quad \text{and } A_- \equiv B_- + iC_-, \tag{37}$$

where B_+, C_+, B_-, and C_- are all real, we then find

$$\begin{aligned}x(t) &= \text{Re}\left[A_+e^{i\omega_0 t}\right] + \text{Re}\left[A_-e^{-i\omega_0 t}\right] \\ &= B_+\cos(\omega_0 t) - C_+\sin(\omega_0 t) + B_-\cos(-\omega_0 t) + C_-\sin(-\omega_0 t) \\ &= (B_+ + B_-)\cos(\omega_0 t) + (-C_+ - C_-)\sin(\omega_0 t),\end{aligned} \tag{38}$$

where we used Eq.(36) in the second line. For simplicity, we can define new constants B and C as

$$B \equiv B_+ + B_- \quad \text{and } C \equiv -C_+ - C_-. \tag{39}$$

Then Eq.(38) becomes

$$x(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) \tag{40}$$

At last, Eq.(40) represents the most general (and completely physical) solution to the simple harmonic equation of motion Eq.(20). We note, as predicted, that this solution is composed of a sine and cosine function which well models our intuition of how the positions of oscillating objects evolve in time.

Constants of Differential Equations: It is important to note that the solution Eq.(40) to the second-order differential equation Eq.(20) has two independent constants (B and C) in the most general solution. This property can be generalized to linear differential equations of arbitrary orders. More generally the differential equation

$$c_n \frac{d^n}{dt^n} x(t) + c_{n-1} \frac{d^{n-1}}{dt^{n-1}} x(t) + \cdots + c_1 \frac{d}{dt} x(t) + c_0 x(t) = 0, \tag{41}$$

where c_k are constants would have n independent solutions. And the most general solution would be parameterized by n constants each of which is associated with a particular initial condition and each of which multiplies one of the independent solutions.

We see this as follows: If we guess the exponential solution $x(t) = ae^{\lambda t}$, we would find that this guess solves Eq.(41) as long as λ satisfies the nth order polynomial

$$c_n \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0 = 0. \tag{42}$$

By the fundamental theorem of algebra, Eq.(72) must have n roots. If its roots are the **unique** values $\lambda_1, \ldots, \lambda_n$, then the general solution to Eq.(41) is

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \cdots a_n e^{\lambda_n t}, \tag{43}$$

where $a_1, \ldots, a_n$ are the constants which are constrained by the initial conditions.

1.6 Initial Conditions

Eq.(40) represents the conclusion of our search for the solution to Eq.(20). It gives us the kinematics of simple harmonic motion and thus affords us with the ability to model the situations depicted in Fig. 1. But Eq.(40) is not completely transparent, in particular it is not clear what B and C represent. It would be much much preferred if we could find a form of Eq.(40) which made the physical parameters of our system completely manifest.

This can be achieved by imposing **initial conditions** on Eq.(40). You might recall that initial conditions define our coordinate and velocity variables at the origin of time in our system. We will start with the most general initial conditions: Let's say our particle starts at a position $x(0) = x_0$ and with a velocity $\dot{x}(0) = v_0$. Given Eq.(40), we then find that B and C are

$$\begin{aligned} x(t) &= B\cos(\omega_0 t) + C\sin(\omega_0 t) \quad \rightarrow \quad x(0) = B = x_0 \\ \dot{x}(t) &= -\omega_0 B\sin(\omega_0 t) + \omega_0 C\cos(\omega_0 t) \quad \rightarrow \quad \dot{x}(0) = \omega_0 C = v_0. \end{aligned} \tag{44}$$

Thus we find Eq.(40) can be written as

$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t), \tag{45}$$

which explicitly includes the initial conditions of the oscillator.

1.7 Amplitude and Phase

We will write Eq.(40) in one more form which highlights two important properties of oscillations. We introduce two new real constants $A_0 > 0$ and ϕ and relate these two constants to B and C through

$$A_0 \cos\phi \equiv \quad \text{and} \quad A_0 \sin\phi \equiv C, \tag{46}$$

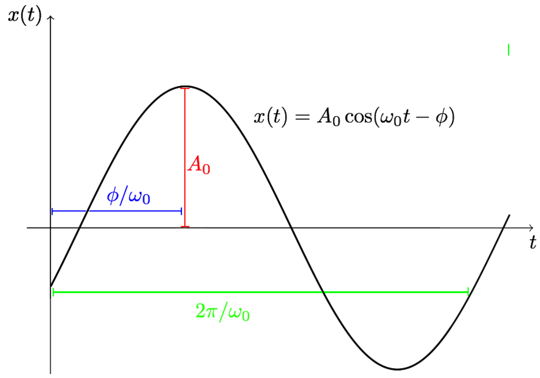


Figure 3: Plot of Eq.(47). The amplitude A_0 represents the maximum displacement from the origin and the phase ϕ defines the time shift ϕ/ω_0 from the $t = 0$ point.

Thus Eq.(40) becomes

$$\begin{aligned} x(t) &= A_0 \cos\phi \cos(\omega_0 t) + A_0 \sin\phi \sin(\omega_0 t) \\ &= A_0 \cos(\omega_0 t - \phi). \end{aligned} \tag{47}$$

Using Eq.(44) and Eq.(46), we can write A_0 and ϕ in terms of the initial conditions x_0 and v_0. Doing so, we have

$$A_0 = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}} \quad \text{and} \quad \phi = \tan^{-1}\frac{v_0}{\omega_0 x_0}. \tag{48}$$

The utility of the form Eq.(47) is that it explicitly includes the **amplitude** A (i.e., the largest displacement) and the **phase** ϕ (i.e., the shit from a $t = 0$ origin) of the periodic motion. From Eq.(48), both of these quantities can be determined from the initial conditions, but they uniquely allow us to plot the position as a function of time. Fig. 3 displays these properties given the equation $x(t)$. What's more, we note that ω_0 defines the **period** T of our oscillation:

$$T = \frac{2\pi}{\omega_0}. \tag{49}$$

T is the amount of time it takes the oscillator to return to the its initial position and velocity. Eq.(49) also explains why we termed ω_0 the angular frequency rather than the frequency; the factor 2π represents 2π "radians" and thus to get a quantity with units of seconds, we need to divide 2π radians by a quantity with units of radians/sec. The definition in Eq.(49) is general for all simpler harmonic oscillator systems, but because ω_0 depends on the parameters of the system the period too will depend on the parameters of the system.

1.8 Energy and Potential Energy

In Eq.(20), we encapsulated the dynamics of simple harmonic oscillator systems into a single equation, but the dynamics of mechanical system can be represented by another physical quantity: the energy. Knowing the energy of the system often allows us to determine kinematical variables in a system without having to work through the equation of motion.

Before we compute the energy for the simple harmonic oscillator, we need to review what enters into the energy. By definition the total **mechanical energy** of a system is defined as

$$E_{\text{tot}} = K + U, \tag{50}$$

where K is kinetic energy of the system and U is the potential energy. The kinetic energy for a single particle system is $\frac{1}{2}m\vec{v}^2$ and for one-dimensional systems reduces to

$$K = \frac{1}{2}m\dot{x}^2. \tag{51}$$

From classical mechanics, we know that any system which has a position dependent potential energy $U(x)$ also experiences a force defined as

$$F(x) = -\frac{dU}{dx}. \tag{52}$$

By the fundamental theorem of calculus, we can use Eq.(52) to write the relationship between force and potential energy as an integral. Doing so we find

$$U(x) = U(x_0) - \int_{x_0}^{x} dx'\, F(x') \qquad \text{[Definition of Potential Energy]} \tag{53}$$

Eq.(53) allows us to compute the potential energy of systems for which the force is given. For the harmonic oscillator, the force is Eq.(15), and thus the potential energy (with $x_{\text{eq}} = 0$) is

$$\begin{aligned} U(x) &= U(x_0) - \int_{x_0}^{x} dx'(-kx) \\ &= U(x_0) + k\int_{x_0}^{x} dx'\, x \\ &= U(x_0) + \frac{1}{2}k(x^2 - x_0^2). \end{aligned} \tag{54}$$

Thus we find

$$U(x) = \frac{1}{2}kx^2, \tag{55}$$

where in the last line of Eq.(54), we imposed the condition $U(x_0) = \frac{1}{2}kx_0^2$. We see then that when the force is linear and directed toward to origin (or, more generally, the equilibrium position), the potential energy is quadratic. We depict this relationship in Fig. 4

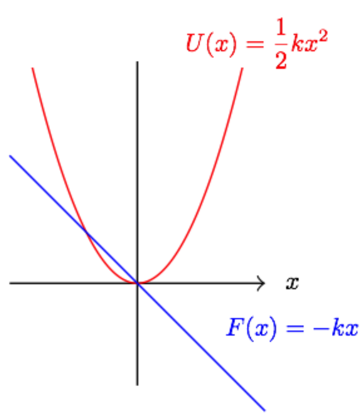


Figure 4: Graphical relationship between force and potential energy of the harmonic oscillator

Thus with Eq.(55) and Eq.(51), we find that the total mechanical energy of the simple harmonic oscillator is

$$E_{\text{tot}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \tag{56}$$

Eq.(56) reveals that for simple harmonic oscillator systems, the total mechanical energy is quadratic in both the velocity and position. But Eq.(56) can be even further simplified. Systems where the force can be written in terms of the potential energy as Eq.(52) satisfy, in microcosm, a fundamental property of our physical world: **conservation of energy**. For systems where conservation of energy applies, the total energy of that system does not change in time, that is

$$\frac{dE_{\rm tot}}{dt} \qquad \text{[Conservation of Energy]}. \tag{57}$$

Our simple harmonic oscillator system satisfies conservation of energy by virtue of Eq.(20). We can show by differentiating Eq.(56) with respect to time and using the chain rule, that $E_{\rm tot}$ is constant in time as long as Eq.(20) is true. We can also demonstrate Eq.(57) another way for the simple harmonic oscillator. Inserting Eq.(45) into Eq.(56), we find

$$\begin{aligned} E_{\rm tot} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\left(-A_0\omega_0\sin(\omega_0 t - \phi)\right)^2 + \frac{1}{2}k\left(A_0\cos(\omega_0 t - \phi)\right)^2 \\ &= \frac{1}{2}m\omega_0^2A_0^2\sin^2(\omega_0 t - \phi) + \frac{1}{2}kA_0^2\cos^2(\omega_0 t - \phi) \\ &= \frac{1}{2}kA_0^2\sin^2(\omega_0 t - \phi) + \frac{1}{2}kA_0^2\cos^2(\omega_0 t - \phi) \\ &= \frac{1}{2}kA_0^2, \end{aligned} \tag{58}$$

where in the last line we used the definition of angular frequency. Thus we see that the total energy of the simple harmonic oscillator is completely defined by the amplitude of the motion. Further, if we substitute Eq.(48) into Eq.(58), we find

$$E_{\rm tot} = \frac{1}{2}k\left(x_0^2 + \frac{v_0^2}{\omega_0^2}\right) = \frac{1}{2}kx_0^2 + \frac{1}{2}mv_0^2, \tag{59}$$

which is Eq.(56) evaluated at $t = 0$. This result is just as we should expect if the energy is conserved: If the energy of the system does not change in time, then the energy at $t = 0$ must be the same as the energy at all subsequent times.

2 The Classical Pendulum

We have just completed a discussion of one canonical system in physics and now we will discuss a related one: the classical pendulum. The classical pendulum is arguably just as widespread as the spring-like oscillations we studied in the last section because *whenever* we have an object hanging from a pivot point in earth's gravitational field, we have a pendulum system. The simples setup for the classical pendulum is shown in Fig. 5.

We have an object of mass m attached to the end of a string of length ℓ. The other end of the string is attached to a pivot point from which the mass-string system hangs. Our objective is the same as before: Determine the dynamics of this system and use those dynamics to determine the kinematics.

We *could* start with Newton's second law for the object of mass m:

$$m\ddot{r}(t) = \vec{F}_{\rm net}. \tag{60}$$

However, it will prove calculationally easier to proceed by using conservation of energy. For the pendulum

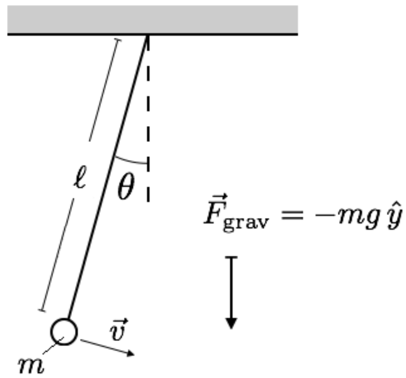


Figure 5: Classical pendulum. An object of mass m is at the end of a string of length ℓ and in a constant gravitational field pointing downward.

system, we define the position coordinate of the object by

$$\vec{r} = \ell \sin\theta\,\hat{x} + \ell(1-\cos\theta)\hat{y}, \tag{61}$$

so that when the object is at $\theta = 0$ it is also at $(x, y) = (0, 0)$. Given the potential energy $U_{\text{grav}} = mgy$, we know that the total mechanical energy of an object of mass m in a gravitational field is

$$E_{\text{tot}} = \frac{1}{2}m\dot{\vec{r}}^2 + mgy. \tag{62}$$

For the purpose of finding the equation of motion of this system, we will express Eq.(62) in terms of angular variables. Given Eq.(61) the velocity of the object is

$$\begin{aligned}\dot{\vec{r}} &= \frac{d}{dt}\ell(\sin\theta\,\hat{x} - \cos\theta\,\hat{y})\\ &= \ell(\cos\theta\,\dot{\theta}\,\hat{x} + \sin\theta\,\dot{\theta}\,\hat{y}),\end{aligned} \tag{63}$$

where we used the chain rule calculation $\frac{d}{dt}\cos\theta(t) = \frac{d}{d\theta}\cos\theta\frac{d\theta}{dt} = -\sin\theta\,\dot{\theta}$. Computing the magnitude squared of this quantity, we find

$$\dot{\vec{r}}^2 = (\ell\cos\theta\,\dot{\theta})^2 + (\ell\sin\theta\,\dot{\theta})^2 = \ell^2\dot{\theta}^2. \tag{64}$$

Also, given Eq.(61) the y coordinate is $y = \ell(1-\cos\theta)$. Thus Eq.(62) becomes

$$E_{\text{tot}} = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta). \quad \text{[Energy of a Pendulum]} \tag{65}$$

Now, we know that the potential energy $U_{\text{grav}} = mgy$ arises from the definition of gravitational force and Eq.(53). Thus, because the external force exerted on our system obeys Eq.(52), our system conserves energy. In other words, the energy in Eq.(65) must satisfy $dE_{\text{tot}}/dt = 0$. Imposing this condition on Eq.(65), we find

$$\begin{aligned}0 &= \frac{dE_{\text{tot}}}{dt}\\ &= \frac{d}{dt}\left[\frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta)\right]\end{aligned}$$

$$\begin{aligned} &= m\ell^2\dot\theta\,\frac{d}{dt}\dot\theta + mg\ell\sin\theta\,\frac{d}{dt}\theta \\ &= m\ell^2\dot\theta\left(\ddot\theta + \frac{g}{\ell}\sin\theta\right), \end{aligned} \tag{66}$$

Since m and ℓ are always non-zero and $\dot\theta$ is generally non-zero, we need the quantity in the parentheses to be zero in order for dE_{tot}/dt to always be zero. Mandating this condition we find

$$\ddot\theta + \frac{g}{\ell}\sin\theta = 0, \quad \text{[Pendulum equation of motion]} \tag{67}$$

which is the equation of motion of the pendulum. At this point you might object to this result. We previously introduced the classical pendulum in Fig. 1 and claimed that it had the same equation of motion as the mass-spring system. And yet, the pendulum equation of motion Eq.(67) seems quite different from the mass-spring equation of motion Eq.(20). For one Eq.(67) is nonlinear in θ and thus does not seem amenable to the "guess-and-check" method we previously employed.

As in many areas in physics, the resolution to this problem comes from an approximation. If θ is sufficiently small, we can use the **Taylor Series** approximation of $\sin\theta$ to write

$$\sin\theta \simeq \theta, \qquad \text{[For } |\theta| \ll 1 \text{ (where } \theta \text{ is in radians)]} \tag{68}$$

Thus, in this **small-angle approximation**, Eq.(67) becomes

$$\ddot\theta(t) + \frac{g}{\ell}\theta(t) = 0. \tag{69}$$

Identifying the angular frequency of the pendulum system as

$$\omega_0 = \sqrt{\frac{g}{\ell}}, \qquad \text{[For small-angle approx.]} \tag{70}$$

we find that Eq.(69) exactly matches Eq.(20). Thus we see that in the small-angle approximation, the equation of motion of the pendulum indeed matches the SHO equation of motion.

2.1 Aside: Taylor Series

In this section, we review the construction of Taylor series. A Taylor series can conceptually be seen as a way to represent a function as a polynomial. In physics, we often deal with transcendental (i.e., logarithmic or exponential) and sinusoidal functions like

$$\ln(x+5), \qquad 10\sin(2t-6), \quad e^{s-2} \tag{71}$$

whose values are determined by numerical tables in textbooks or online. However, when we first learn algebra, we deal with polynomial expression such as

$$s+1, \qquad x^3+2x+1, \qquad t^2-2t. \tag{72}$$

The presumption of the Taylor series is that for certain domains of the functions $\ln x$, e^x and $\sin x$ (and virtually any other function), we can approximate the function with a polynomial like those in Eq.(72). Pursuing this presumption, say we have a general function $f(x)$ which can be approximated by a polynomial for x near $x = 0$. Writing $f(x)$ as an arbitrary polynomial of x, we then make the claim

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{k=0}^{\infty} a_kx^k, \tag{73}$$

where the a_ks are the set of coefficients which make Eq.(73) true. Given Eq.(73), we can automatically determine one of these coefficients. Setting $x = 0$ on both sides we find

$$a_0 = f(x). \tag{74}$$

Now, if Eq.(73) is true, we should be able to find a similar relationship for the derivatives of $f(x)$. Differentiating Eq.(73) once, we have

$$\frac{d}{dx}f(x) = \frac{d}{dx}\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k \frac{d}{dx}x^k = \sum_{k=0}^{\infty} a_k k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots . \tag{75}$$

Setting $x = 0$ in Eq.(75), we find

$$a_1 = f'(0). \tag{76}$$

We can repeat this pattern for the higher derivatives of $f(x)$. In general, we would find that the a_n are given by

$$a_n = \frac{1}{n!}f^{(n)}(0), \tag{77}$$

where $f^{(n)}(0)$ stands for the "nth derivative of $f(x)$ which is then evaluated at $x = 0$". Therefore Eq.(73) becomes

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2}f''(0)\,x^2 + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k. \tag{78}$$

If the summation on the right-hand-side of Eq.(78) converges to a finite value then it serves as a good representation of $f(x)$. We could generalize this discussion to consider the function $f(x)$ near the point $x = a$ rather than $x = 0$. Doing so, we find that the Taylor series becomes

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k, \quad \text{[Taylor Series]} \tag{79}$$

which again is only valid so long as the summation converges to a finite value.

Below we list a few Taylor Series (near $x = 0$) for important functions we will use in class. We include the domain of x in which these Taylor Series are valid.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \qquad \text{valid for all } x \tag{80}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^k\frac{x^{2k+1}}{(2k+1)!} + \cdots \qquad \text{valid for all } x \tag{81}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^k\frac{x^{2k}}{(2k)!} + \cdots \qquad \text{valid for all } x \tag{82}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{k}x^k + \cdots \quad \begin{cases} n \text{ is an integer} & \text{valid for all } x \\ \text{otherwise} & \text{valid for } |x| < 1 \end{cases} \tag{83}$$

It is worth noting that Eq.(83) is both the Taylor Series and binomial expansion of $(1+x)^n$. Indeed, we can think of Taylor Series as a sort of binomial expansion for an arbitrary function.

2.1.1 Period of Pendulum

We should mention one more thing. Given Eq.(70) and the definition of period in Eq.(49), we find that the period of the pendulum (in the small-angle approximation) is

$$T = 2\pi\sqrt{\frac{\ell}{g}}, \qquad \text{[For small-angle approx.]} \tag{84}$$

Eq.(84) is consistent with the observational fact that the period of a pendulum is independent of the mass attached to the end of the string. Moreover, T is inversely proportional to the gravitational acceleration g and thus decreases with increasing strength of the gravitational field at the surface of a planet. Therefore swinging on a swing set would take a longer time on the moon and a short time on Saturn (assuming, of course, either was habitable).

Of course, Eq.(84) is only true for $|\theta| \ll 1$. For more general θ we would need the conservation of energy condition Eq.(65) to compute the exact period.

3 Oscillations around Stable Equilibria

In the previous two sections, we discovered that the mass-spring system and the pendulum, two systems which are quite different, have equations of motion Eq.(17) and Eq.(69), respectively which are identical in form. The similarity between the two equations of motion can lead us toward a more general understanding of simple harmonic oscillator systems. We can ask the question, does this similarity suggest some wider class of systems which obey Eq.(20)?

It certainly does! In fact all energy-conserving systems whose position coordinate remains near a stable equilibrium have dynamics defined by Eq.(20). A **stable equilibrium** is defined as a position where the net forces acting on the system are zero, and where deviations from that position result in forces which push the object back towards the equilibrium position.

These ideas can be clarified if we express them mathematically. Say we have a particle of mass m which is acted upon by the net-force F. We will take F to be a function of the position coordinate x. By Eq.(52), (and assuming the system conserves energy) this force must be related to the potential energy $U(x)$ of our system through the equation

$$F(x) = -\frac{dU}{dx}. \tag{85}$$

Next, Let us say that our mass is at a position x_{eq} where $F(x) = 0$:

$$F(x = x_{\text{eq}}) = 0 \quad \text{[Definition of equilibrium position]}. \tag{86}$$

Such a position is called an equilibrium position (but not necessarily a *stable* equilibrium) because the net force acting on it is zero. Now, let us move our particle slightly from this equilibrium position by an amount δx so that the coordinate x becomes

$$x = x_{\text{eq}} + \delta x, \tag{87}$$

where $\delta x \ll x_{\text{eq}}$. What is the approximate force, i.e., the approximation of Eq.(85), near this equilibrium position? Beginning with the potential energy, we can perform a Taylor expansion about x_{eq}. Doing so we find

$$U(x) = U(x_{\text{eq}} + \delta x) = U(x_{\text{eq}}) + U'(x_{\text{eq}})\,\delta x + \frac{1}{2}U''(x_{\text{eq}})\,\delta x^2 + \mathcal{O}\left((\delta x)^3\right), \tag{88}$$

where $\mathcal{O}\left((\delta x)^3\right)$ stands in for terms proportional to δx^3 or higher powers of δx. By Eq.(85) and Eq.(86), we know that $U'(x_{\text{eq}}) = 0$, and so this potential energy reduces to

$$U(x) \simeq U(x_{\text{eq}}) + \frac{1}{2}U''(x_{\text{eq}})\,\delta x^2, \tag{89}$$

where we substituted in $\delta x = (x - x_{\text{eq}})$ and dropped the higher order terms of δx. Now, we can compute the near-equilibrium approximation of Eq.(85). Given that x_{eq} is a constant and any function evaluated at $x = x_{\text{eq}}$ is also a constant, we find

$$F(x) = -U'(x) \simeq -U''(x_{\text{eq}})(x - x_{\text{eq}}). \tag{90}$$

Now with the near equilibrium force Eq.(90) we can determine the near equilibrium dynamics of this system. Using Newton's Second law we find

$$m\ddot{x} = F(x) \simeq -U''(x_{\text{eq}})(x - x_{\text{eq}}). \tag{91}$$

Moving the potential energy term to the left hand side and dividing the entire equation by m, we then obtain

$$\ddot{x} + \frac{U''(x_{\text{eq}})}{m}(x - x_{\text{eq}}) \simeq 0. \tag{92}$$

We previously stated that the condition for a position x_{eq} to be an equilibrium is that $F(x)$ (or $U'(x)$) needs to be zero at that position. Now in order for this position to be a *stable* equilibrium, we need the additional condition

$$U''(x_{\text{eq}}) > 0 \quad \text{[Condition for a stable equilibrium position]}. \tag{93}$$

If $U(x_{\text{eq}}) > 0$, then we can define the angular frequency

$$\omega_0 = \sqrt{\frac{U''(x_{\text{eq}})}{m}}, \tag{94}$$

and Eq.(92) would become

$$\ddot{x} + \omega_0^2(x - x_{\text{eq}}) \simeq 0, \tag{95}$$

or, with $x - x_{\text{eq}} = \delta x$,

$$\ddot{\delta x}(t) + \omega_0^2\,\delta x(t) \simeq 0. \tag{96}$$

We use the "approximately equal" symbol $\simeq$ because Eq.(96) is only valid for δx sufficiently small. Eq.(96) is identical in form to Eq.(20) and thus all our previous results (amplitude, period, phase, energy properties) for the simple harmonic oscillator apply to the system defined by Eq.(96) as well. Thus, we can understand why the simple harmonic motion is so ubiquitous. The world contains many different systems which are at or near a stable equilibrium, and when these systems are perturbed from that equilibrium they must undergo motion defined by Eq.(96).

To review, the two conditions required in order for a point $x = x_1$ to be a stable equilibrium (and thus for it to exhibit simple harmonic motion defined by Eq.(96)) are

- First derivative is zero: $U'(x_1) = 0$
- Second derivative is positive: $U''(x_1) > 0$

More formally a stable equilibrium of a system is associated with a **local minimum of the potential energy**. A function $U(x)$ can have multiple stable equilibria and around each equilibrium the particle would undergo simple harmonic oscillations. This is because around each equilibrium we can approximate $U(x)$ as a quadratic function (See Fig. 6).

The results of this section are so important they deserve to be stated as a theorem of mathematical physics.

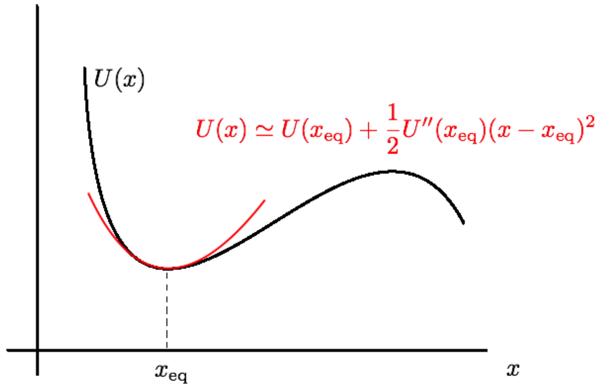


Figure 6: A general function $U(x)$ can be approximated as a quadratic function near a stable equilibrium $x_{\rm eq}$.

SHO theorem of physics: All energy-conserving physical systems with position coordinates sufficiently close to a stable equilibrium have an equation of motion of the form

$$\ddot{\delta x}(t) + \omega_0^2\, \delta x(t) \simeq 0, \tag{97}$$

where $\delta x(t)$ is the position coordinate's deviation from the equilibrium $x_{\rm eq}$, and ω_0 is the angular frequency. Given the potential energy $U(x)$ this angular frequency is

$$\omega_0 = \sqrt{\frac{U''(x_{\rm eq})}{m}}, \tag{98}$$

where $x_{\rm eq}$ is the coordinate where $U'(x) = 0$ and m is the mass of the system.

This theorem is why we can have the three rather different systems of Fig. **??** which nevertheless have the same equation of motion.

We have not completed our fist foray into the physics of oscillating systems. In the next lecture notes we will complicate this story by relaxing our ideal ”energy-conserving” assumption and considering what happens when oscillating systems are subject to fluid drag.

3.1 Interaction of Atoms

In this section, we provide an example of the above outlined procedure and thus show that we find small oscillations in systems which are near equilibrium

When atoms (or molecules) are separated by distances of about an angstrom (i.e., 10^{-10} m), they interact through what is known as the **Lennard-Jones potential**. The Lennard-Jones potential gives the potential energy of interaction between the two atoms as a function of the distance between them. For example, two Argon atoms separated by a distance r would interact with the potential energy

$$U(r) = \epsilon\left[\left(\frac{r_{\rm m}}{r}\right)^{12} - 2\left(\frac{r_{\rm m}}{r}\right)^{6}\right] \tag{99}$$

where ϵ and $r_{\rm m}$ (both of which are positive) have the units of energy and distance, respectively, and both parameters can be experimentally determined. We want to answer two questions:

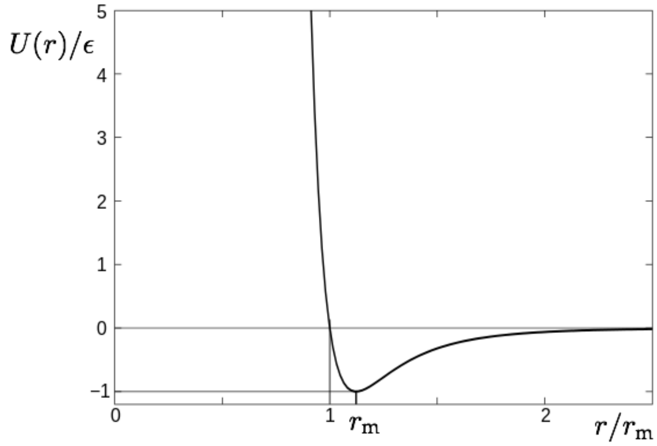


Figure 7: Image from Wikipedia entry on Lennard-Jones

1. At what value of r is the force between the atoms zero? (This value of r is what we call the equilibrium position.)
2. Second, what is the (angular) frequency of small oscillations about this equilibrium position?

To answer the first question we use the definition of force in terms of potential energy to find

$$F(r) = -\frac{dU}{dr} = -\epsilon\left[12\left(\frac{r_\mathrm{m}}{r}\right)^{11} - 12\left(\frac{r_\mathrm{m}}{r}\right)^{5}\right]\left(-\frac{r_\mathrm{m}}{r^2}\right), \tag{100}$$

where we used the chain rule and $\frac{d}{dr}(r_\mathrm{m}/r) = -r_\mathrm{m}/r^2$. To find the value of r where this force is zero (i.e., where the interacting atoms are in equilibrium), we set this quantity to zero and solve for r. Doing so we obtain

$$\begin{aligned} 0 &= F(r) \\ &= \epsilon\left[12\left(\frac{r_\mathrm{m}}{r}\right)^{11} - 12\left(\frac{r_\mathrm{m}}{r}\right)^{5}\right]\left(\frac{r_\mathrm{m}}{r^2}\right) \\ \left(\frac{r_\mathrm{m}}{r}\right)^{11} &= \left(\frac{r_\mathrm{m}}{r}\right)^{5} \\ \left(\frac{r_\mathrm{m}}{r}\right)^{6} &= 1. \end{aligned} \tag{101}$$

The last equality implies that the solution is $r = r_\mathrm{m}$. Thus we find that the force $F(r) = -U'(r)$ is 0 when $r = r_\mathrm{m}$.

Now, in order to determine whether there are small oscillations about this point, we can either expand Eq.(99) about $r = r_\mathrm{m}$ and keep the quadratic term, or we can take the second derivative of Eq.(99) and (according to Eq.(**??**)) check that the result is zero. We will do the latter. Taking the second derivative of

Eq.(99), we find

$$\begin{aligned}
\frac{d^2}{dr^2}U(r) &= \frac{d^2}{dr^2}\left\{\epsilon\left[\left(\frac{r_{\rm m}}{r}\right)^{12} - 2\left(\frac{r_{\rm m}}{r}\right)^{6}\right]\right\} \\
&= \frac{d}{dr}\left\{\epsilon\left[12\left(\frac{r_{\rm m}}{r}\right)^{11} - 12\left(\frac{r_{\rm m}}{r}\right)^{5}\right]\left(-\frac{r_{\rm m}}{r^2}\right)\right\} \\
&= \epsilon\left[132\left(\frac{r_{\rm m}}{r}\right)^{10} - 60\left(\frac{r_{\rm m}}{r}\right)^{7}\right]\left(-\frac{r_{\rm m}}{r^2}\right)^2 \\
&\quad + \epsilon\left[12\left(\frac{r_{\rm m}}{r}\right)^{11} - 12\left(\frac{r_{\rm m}}{r}\right)^{5}\right]\left(\frac{2r_{\rm m}}{r^3}\right). \qquad (102)
\end{aligned}$$

Taking $r = r_{\rm m}$, we find that the second term in the last line of Eq.(102) is zero. Thus only the first term remains and we obtain

$$U''(r = r_{\rm m}) = \epsilon\left[132\left(\frac{r_{\rm m}}{r_{\rm m}}\right)^{10} - 60\left(\frac{r_{\rm m}}{r_{\rm m}}\right)^{7}\right]\left(-\frac{r_{\rm m}}{r_{\rm m}^2}\right)^2 = +\frac{72\epsilon}{r_{\rm m}^2}. \qquad (103)$$

With $\epsilon > 0$, we indeed see that Eq.(**??**) is satisfied and thus this system is capable of small oscillations about $r = r_{\rm m}$. We can thus conclude that Eq.(99), for r close to $r_{\rm m}$, satisfies the approximation

$$\begin{aligned}
U(r) &= \epsilon\left[\left(\frac{r_{\rm m}}{r}\right)^{12} - 2\left(\frac{r_{\rm m}}{r}\right)^{6}\right] \\
&= U(r_{\rm m}) + U'(r_{\rm m})(r - r_{\rm m}) + \frac{1}{2}U''(r)(r - r_{\rm m})^2 + \cdots \\
&= -\epsilon + \frac{36\epsilon}{r_{\rm m}^2}(r - r_{\rm m})^2 + \cdots . \qquad (104)
\end{aligned}$$

where we used Eq.(**??**) to obtain the second line, and we used Eq.(99), Eq.(100) and Eq.(103) to obtain the last line.

We therefore see that the potential energy Eq.(99) can be approximated by a quadratic function of r when r is near the equilibrium position. This indicates that for r near $r_{\rm m}$, the two interacting atoms can undergo simple harmonic motion.

Now we want to know the frequency of the oscillation. If we treat the position of one of the atoms as fixed (let's say one atom has a much larger mass than the other atom[5]), then the other atom of mass m has the equation of motion

$$m\ddot{r} = F(r), \qquad (105)$$

where $\ddot{r}$ is the acceleration of the atom and $F(r)$ is the force from the potential Eq.(99). Given the approximate potential energy Eq.(104), and the definition of the derivative we find that this equation of motion can be approximated as

$$m\ddot{r} = F(r) = -\frac{d}{dr}\left[-\epsilon + \frac{36\epsilon}{r_{\rm m}^2}(r - r_{\rm m})^2 + \cdots\right] = -\frac{72\epsilon}{r_{\rm m}^2}(r - r_{\rm m}) + \cdots . \qquad (106)$$

If we then define R as

$$R \equiv r - r_{\rm m}, \qquad (107)$$

[5]This is not the standard way this potential is studied, but we make this simplification in order to not have to introduce the concept of reduced mass.

then the time derivative of R is equal to the time derivative of r (because r_m is constant):

$$\frac{d}{dt}(R) = \frac{d}{dt}(r - r_\mathrm{m}) = \frac{d}{dt}r, \tag{108}$$

and similarly $\frac{d^2}{dt^2}r = \frac{d^2}{dt^2}R$. Therefore the approximate equation of motion becomes

$$m\ddot{R} \simeq -\frac{72\epsilon}{r_\mathrm{m}^2}R + \cdots \,, \tag{109}$$

or, dividing by m, moving all terms to one side, and dropping the higher order terms

$$\ddot{R} + \omega_0^2 R = 0, \tag{110}$$

where we defined

$$\omega_0 \equiv \sqrt{\frac{72\epsilon}{m r_\mathrm{m}^2}}. \tag{111}$$

Eq.(110) is the equation for simple harmonic motion. Therefore, we see that the potential energy Eq.(99) produces simple harmonic motion when r is close to r_m and that this motion has an angular frequency of Eq.(111).

Solving Eq.(110), we find that the atoms oscillate about their equilibrium position according to

$$R(t) = R_0 \cos(\omega_0 t) + \frac{V_0}{\omega_0}\sin(\omega_0 t), \tag{112}$$

where R_0 is the initial value of $R(t)$ and V_0 is the initial value of $\dot{R}(t)$.