Lecture 04: Damped Oscillations

In these notes, we complicate our previous discussion of the simple harmonic oscillator by considering the case in which energy is not conserved. Specifically we consider how the dynamics and kinematics of the oscillator change when we subject it to a velocity-dependent damping force.

1 Decreasing Amplitude

We previously claimed to have found a mathematical way to characterize systems which oscillate about a stable equilibrium, but our model was not as accurate as it could be. If you have ever setup a pendulum and allowed it to swing for a few periods, you would notice that its angular position does not exactly look like the constant oscillations of a sinusoid. Rather, we would find that the amplitude of the swing decreases over time until the pendulum stops moving (See Fig. 1).

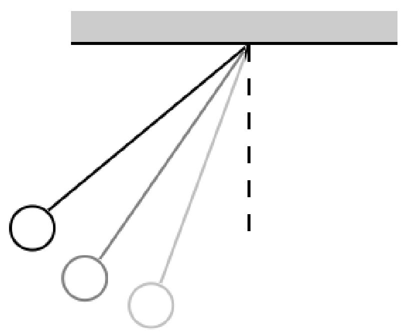


Figure 1: Damped pendulum: Pendulums in real life are always acted upon by air drag (in addition to the frictional forces at their pivots) and thus they do not conserve energy. In the figure, the successive shadows depict successive amplitudes of the pendulum swing.

The reason this happens is that, unlike the ideal world of physical models, the real world is somewhat messy. As the pendulum swings, the massive object bumps into molecules in the air and this collective bumping works to rob the pendulum of its initial energy. Over time these molecular collisions dissipate all of the energy of the pendulum so that its amplitude of oscillations get smaller and smaller.

Consequently the total mechanical energy of a pendulum (and pretty much any real oscillating system) is not actually conserved and the associated equation of motion of is not$\ddot{\theta}+\omega_0^2\theta = 0$. To find the correct equation of motion we would need to find a way to model oscillatory phenomena which decrease in amplitude due to air drag.

Framing Question

How do we model oscillatory phenomena in which air drag causes a decrease in oscillation amplitude?

1.1 Drag and general Damping Forces

To achieve our objective of finding a more accurate model for oscillatory phenomena, we need to first find the correct Newton's second law equation for such systems. Thus we need to better determine the forces acting on our oscillating object. When previously analyzing simple harmonic oscillations, we only considered position dependent and energy-conserving forces, but if we know that real oscillations do not conserve mechanical energy we need to consider the forces responsible for this non-conservation.

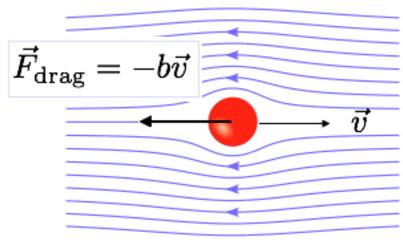


Figure 2: Drag force on a spherical object moving through a liquid. The drag force opposes the direction of motion and is proportional to the speed. Figure adapted from Wikipedia.

In mechanics, such forces are generally termed **frictional forces**. You have likely encountered frictional forces like static friction and kinetic friction in a mechanics class before. These forces arise when two solid objects are in contact with and slide past one another, but this is not the type of frictional force we're interested in.

Rather we're interested in the case when a solid object is moving through a liquid or gaseous medium. This movement is associated with a particular type of frictional force called a **drag force**. For example, when a spherical ball moves at low speeds[1] through a liquid medium the drag force exerted on the ball is

$$\vec{F}_{\text{drag}} = -b\vec{v}, \tag{1}$$

where b is a constant which depends on the properties of the medium and the geometric properties of the ball. Eq.(1) also well models the force exerted on objects moving through air, so we will use it as our main drag force in what follows.

2 Damped Oscillations

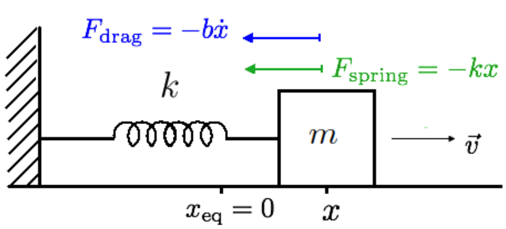


Figure 3: Damped Harmonic Oscillator

With the force of air drag (for sufficiently low velocities) given by Eq.(1) we can now analyze harmonic oscillator motion subject to a velocity dependent drag force. We will consider the one-dimensional mass-

[1]Defined by a quantity termed the Reynolds Number. At higher speeds, the drag force is a quadratic function of speed.

spring system with the knowledge that this analysis can be generalized to any other oscillating system. We will also take our stable equilibrium position to be at $x = 0$ for simplicity.The system is depicted in Fig. 3. Including the drag force, the total force exerted on the object is

$$F_{\text{net}} = F_{\text{drag}} + F_{\text{spring}} = -b\dot{x} - kx. \tag{2}$$

For most problems we will take b as given. By Newton's second law, the equation of motion for the mass is therefore

$$m\ddot{x} = F_{\text{net}} = -b\dot{x} - kx, \tag{3}$$

or

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{4}$$

where we defined

$$\gamma = \frac{b}{2m} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}. \tag{5}$$

Eq.(4) is the desired equation of motion for harmonic motion with air drag. It models what is known as **damped harmonic oscillations**, and is more realistic than the case where b is assumed to be zero. It can thus be readily applied to most every-day oscillating systems provided they can be defined one-dimensionally.

With our new equation of motion Eq.(4), our next task is to solve it and obtain the explicit kinematics (that is, positions and velocities) for the damped harmonic oscillator. Fortunately, we see that Eq.(4) is linear in $x(t)$ and thus is amenable to the "exponential guess" method we employed to solve the simple harmonic oscillator equation of motion.

Recalling the exponential guess method, the steps are to

1. Guess an exponential solution of the form $Ae^{\alpha t}$
2. Derive the condition α must satisfy in order for $Ae^{\alpha t}$ to be a solution of the equation of motion
3. Solve the condition for values of $\alpha_1, \ldots, \alpha_n$
4. If the α_i are unique[2], write general solution as $x(t) = A_1e^{\alpha_1 t} + A_2e^{\alpha_2 t} + \cdots + A_ne^{\alpha_n t}$.

We employ these steps as follows. Assuming an exponential solution of the form $x = Ae^{\alpha t}$, we find Eq.(4) becomes

$$A\left(\alpha^2 + 2\gamma\alpha + \omega_0^2\right)e^{\alpha t} = 0, \tag{6}$$

thus the condition α must satisfy is

$$\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0. \tag{7}$$

Solving Eq.(7), we find the two solutions

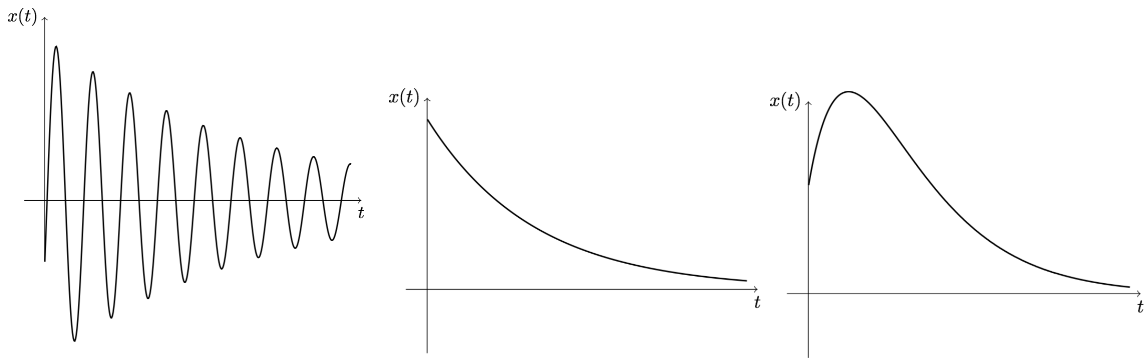
$$\alpha_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{8}$$

These two solutions of α define the two independent solutions of Eq.(4). And from these solutions we find there are three ways γ and ω_0 can be related which lead to distinct forms of $\alpha_\pm$.

- $\omega_0 > \gamma$ (Weak damping)
- $\omega_0 < \gamma$ (Strong damping)
- $\omega_0 = \gamma$ (Critical damping),

we explore each of these cases below including their associated kinematic predictions.

[2]If the α_i are not unique the general solution will be different. See "Critically Damped" section.



(a) Underdamped (b) Overdamped (c) Critically damped

Figure 4: The three possible cases of damped harmonic oscillation

2.1 Underdamped

For the case where $\omega_0 > \gamma$, the spring force dominates the drag force, and the system still exhibits oscillations. We call such a scenario, ”underdamped” harmonic motion. Defining

$$\Omega \equiv \sqrt{\omega_0^2 - \gamma^2}, \tag{9}$$

we find that the general solution in this case is

$$x(t) = A_+ e^{\alpha_+ t} + A_- e^{\alpha_- t} = e^{-\gamma t}\left(A_+ e^{i\Omega t} + A_- e^{-i\Omega t}\right). \tag{10}$$

The quantity in the parentheses of Eq.(10) is complex and yet we know that the solution $x(t)$ must be real. Thus to find the real solution we take the real part of the quantity in the parentheses—in the same way we did so for the simple harmonic oscillator. We then find that the general solution for underdamped motion can be written as

$$x(t) = e^{-\gamma t}\left(A_+ \cos(\Omega t) + A_- \sin(\Omega t)\right). \tag{11}$$

But the most useful form is one where the exponential factor multiplies a single sinusoid:

$$x(t) = e^{-\gamma t} A_0 \cos(\Omega t - \phi), \tag{12}$$

where A_0 and ϕ are determined by initial conditions. This predicted position as a function of time is depicted in Fig. 4a. We note that $x(t)$ exhibits oscillations with a definitive period, but the amplitude of the oscillation decays away exponentially.

2.2 Overdamped

For the case where $\gamma > \omega_0$, the drag force dominates over the spring force and we have what is known as ”overdamped” motion. In this case, we can define

$$\Gamma \equiv \sqrt{\gamma^2 - \omega_0^2}, \tag{13}$$

and the general solution to Eq.(4) becomes

$$x(t) = A_+ e^{\alpha_+ t} + A_- e^{\alpha_- t} = e^{-\gamma t}\left(A_+ e^{\Gamma t} + A_- e^{-\Gamma t}\right) \tag{14}$$

Because $\gamma > \Gamma$, Eq.(14) represents a solution in which $x(t)$ decays to zero over time. Thus the over-damped harmonic oscillator is not truly an oscillator at all. The drag force so impedes the spring force that the oscillations no longer occur. This motion is depicted in Fig. 4b.

2.3 Critically Damped

When $\omega_0 = \gamma$, then the two roots of Eq.(7) are not unique, and so we cannot write the general solution as a linear combination of two unique solutions. Instead, whenever we're solving a homogeneous linear differential equation and we find that a root repeats, then general solution consist of a polynomial multiplying the exponential of the repeating root. For the case Eq.(4), for $\omega_0 = \gamma$, the theory of differential equations tells us that the general solution is

$$x(t) = (A + Bt)e^{-\gamma t}. \tag{15}$$

You don't have to take my word for this: We can take Eq.(15) as a guess and check it against Eq.(4) for $\gamma = \omega_0$. This predicted position is depicted in Fig. 4c. We see in the figure that critically damped motion, like overdamped motion, does not exhibit any oscillations. It can also (like critically damped motion) exhibit changes in direction.

3 Properties of Underdamped Motion

We will now focus on underdamped motion, because seeing as it contains both exponential decay and sinusoidal oscillations it is the most mathematically interesting of the cases explored above. In the real world, unless they are specifically prepared to be otherwise, all oscillations are under damped. Consequently it would prove useful to define a specific quantity which determines the extent to which an oscillatory system is damped. This leads us to a new framing question.

Question

How can we characterize the extent to which a system is underdamped?

An obvious guess is that we can characterize the amount of damping in the same way we did so above: By the ratio ω_0/γ. This turns out to be correct, but we will get to this answer again by considering the energy and power of the oscillatory system.

3.1 Energy and Power

For the simple harmonic oscillator, one of its fundamental properties was that its total mechanical energy was conserved. This is not so for the damped harmonic oscillator. We can see this by starting with the equation for the total mechanical energy of a spring oscillator system:

$$E_{\text{tot}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \tag{16}$$

We note that we don't include a drag force term in Eq.(16) because such a term is neither related to the kinetic nor the potential energy. Differentiating Eq.(16) with respect to time we find

$$\begin{aligned}
\frac{dE_{\text{tot}}}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) \\
&= m\dot{x}\ddot{x} + kx\dot{x} \\
&= m\dot{x}\left(\ddot{x} + \omega_0^2 x\right)
\end{aligned}$$

$$= -2m\gamma\,\dot{x}^2, \tag{17}$$

where we used Eq.(4) in the final line. Because m, γ, and $\dot{x}^2$ are exclusively positive quantities, we thus see that the damped harmonic oscillator decreases in time until it reaches zero. This is consistent with what we observe in oscillations whose amplitude decrease to zero.

Eq.(17) has a specific interpretation in terms of another dynamical quantity. Since the time rate change of energy is power, the mechanical power P supplied to (or taken away from a system) is given by

$$P = \frac{dE_{\text{tot}}}{dt}. \tag{18}$$

When Eq.(18) is positive, then power is being added to the system; when it is less than zero, power is being taken away from the system; when it is zero, energy is conserved. Eq.(17) thus indicates that the drag forces are responsible for robbing the oscillator of power until its energy has decayed to zero.

How exactly does this energy decay to zero? We can answer this question by using Eq.(17) to obtain an approximate equation for how the total energy evolves in time. In order to find this equation, we make the "very weak damping" approximation

$$\gamma \ll \omega_0 \simeq \Omega \qquad \text{[Very Weak Damping]}. \tag{19}$$

Where we used Eq.(9), to also say $\gamma \ll \Omega$. Computing the total mechanical energy Eq.(16), for the solution Eq.(12) we obtain

$$\begin{aligned} E_{\text{tot}} &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}m\Omega^2\Big(-\frac{\gamma}{\Omega}e^{-\gamma t}A_0\cos(\Omega t-\phi) - e^{-\gamma t}A_0\sin(\Omega t-\phi)\Big)^2 + \frac{1}{2}ke^{-2\gamma t}A_0^2\cos^2(\Omega t-\phi). \end{aligned} \tag{20}$$

Since $\gamma \ll \Omega$, we can approximately ignore the first term in the parentheses of Eq.(20). Thus the total energy has the form

$$\begin{aligned} E_{\text{tot}} &\simeq \frac{1}{2}m\Omega^2e^{-2\gamma t}A_0^2\sin^2(\Omega t-\phi) + \frac{1}{2}ke^{-2\gamma t}A_0^2\cos^2(\Omega t-\phi) \\ &\simeq \frac{1}{2}m\omega_0^2e^{-2\gamma t}A_0^2\sin^2(\Omega t-\phi) + \frac{1}{2}ke^{-2\gamma t}A_0^2\cos^2(\Omega t-\phi) \\ &= \frac{1}{2}ke^{-2\gamma t}A_0^2, \end{aligned} \tag{21}$$

where in the second line we used the "very weak damping" approximation $\Omega \simeq \omega_0$. By Eq.(21), we find $E_{\text{tot}}(0) \simeq \frac{1}{2}kA_0^2$ much like the simple harmonic oscillator, and thus we can write

$$E_{\text{tot}}(t) \simeq E_{\text{tot}}(0)e^{-2\gamma t}. \tag{22}$$

Therefore, we find that for very weakly damped oscillations, the total mechanical energy of the harmonic oscillator decays away exponentially in time.

3.2 Quality Factor

We can use the behavior given in Eq.(22) to develop a quantitative measure of how well an oscillator stores energy. To this end we seek to determine how much energy is lost by a damped oscillator as it proceeds through an oscillation cycle. Thus, we compute the ratio

$$\frac{\text{Energy stored in oscillator}}{\text{Energy lost per radian of oscillation}} \tag{23}$$

If no energy is lost per radian of oscillation, then this ratio would be infinite and the oscillator would exhibit no damping. If all the energy of the oscillator is lost over a single radian of oscillation, then this ratio would be to 1 and the oscillator would be virtually incapable of storing energy. How does this definition compare to our previously defined parameters? By the definition of angular frequency (in units of radians per second), a single radian of oscillation is defined by t_1 where

$$1 \text{ radian} = \Omega t_1 \simeq \omega_0 t_1. \tag{24}$$

We note that radians are a dimensionless quantity which are only specified here for precision. Thus, we find that the time to cover a single radian of oscillation is $t_1 \simeq 1/\omega_0$. Over this time period, the energy lost in the oscillator is

$$\begin{aligned}\Delta E_{\text{lost}} &= E_{\text{tot}}(0) - E_{\text{tot}}(t_1) \\ &= E_{\text{tot}}(0)(1 - e^{-2\gamma t_1}) \\ &\simeq E_{\text{tot}}(0)(2\gamma t_1) \qquad \text{[Taylor expansion of exponential]} \\ &\simeq E_{\text{tot}}(0)\frac{2\gamma}{\omega_0}.\end{aligned} \tag{25}$$

Therefore, by Eq.(23) the desired ratio is

$$\frac{E_{\text{tot}}(0)}{\Delta E_{\text{lost}}} \simeq \frac{\omega_0}{2\gamma} \tag{26}$$

We define the quantity $\omega_0/2\gamma$ as the quality factor Q of an underdamped oscillator:

$$Q \equiv \frac{\omega_0}{2\gamma} \qquad \text{[Definition of Quality Factor].} \tag{27}$$

We see then this quality factor definition is consistent with our previous discussion of the underdamped oscillator. For $Q \sim 1$, we have $\omega_0 \sim 2\gamma$, and the oscillator is thus close to being critically damped. When an oscillator is critically damped it is no longer truly an oscillator and thus cannot store energy[3]. Alternatively, for $Q \gg 1$, we have $\omega_0 \gg 2\gamma$, and the oscillator is hardly damped at all and, consequently, is capable of oscillating for many periods before it decays to zero. Moreover, given that $\omega_0 = \sqrt{k/m}$ and $2\gamma = b/m$ for the mass-spring system, we find that the quality factor for this system is

$$Q_{\text{mass-spring}} = \frac{\sqrt{mk}}{b}. \tag{28}$$

Therefore under the assumption of fixed b, we can increase the quality factor of the mass-spring oscillator by increasing its mass or increasing the strength of the spring.

4 Damped Pendulum

We started these notes with the case of a pendulum moving under the forces of air drag. Given our discussion of damped simple harmonic motion, we can now write the equation of motion for such a system. For the pendulum, the velocity of the object at it's end is given by $v = \ell\dot{\theta}$. Therefore the drag force exerted on the pendulum bob is

$$F_{\text{drag}} = -b\ell\dot{\theta}. \tag{29}$$

[3]Given our previous definitions of critical damped motion, if Q is only defined for underdamped oscillators then it is bounded below by $1/2$; less than $1/2$ corresponds to overdamped oscillators.

Noting that that power[4] is given by

$$P = \vec{F}_{\text{ext}} \cdot \vec{v} = \frac{dE_{\text{tot}}}{dt}, \tag{30}$$

and that the only external external force on the pendulum is the drag force, we find

$$\begin{aligned} \vec{F}_{\text{ext}} \cdot \vec{v} &= \frac{dE_{\text{tot}}}{dt} \\ F_{\text{drag}} \times \ell\dot{\theta} &= \frac{d}{dt}\left[\frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta)\right] \\ -b\ell^2\dot{\theta}^2 &= m\ell^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{\ell}\sin\theta\right), \end{aligned} \tag{31}$$

Dividing the entire equation by $m\ell^2\dot{\theta}$ and moving all terms to one side, we find the equation of motion

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2 \sin\theta = 0, \tag{32}$$

where $\gamma = b/2m$ and $\omega_0^2 = g/\ell$. Eq.(33) is the equation of motion for the damped pendulum. Like the equation of motion for the pendulum itself, this equation is difficult to solve without approximation. In the small angle approximation we can take $\sin\theta \simeq \theta$, to obtain the approximate equation of motion

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2\theta \simeq 0. \tag{33}$$

Now, although pendulum systems are almost always damped, their damping is often so slight that we can consider them very weakly damped. Thus we can calculate the quality factor of the pendulum to be

$$Q_{\text{pendulum}} = \frac{\omega_0}{2\gamma} = \frac{m}{b}\sqrt{\frac{g}{\ell}}. \tag{34}$$

Thus, we see that shorter pendulums with heavier bobs store energy better than longer pendulums with lighter bobs.

Power = Force $\times$ Velocity: We know from basic classical mechanics, that when an external force $\vec{F}_{\text{ext}}$ acts upon a particle, it does work W on the particle by an amount equivalent to the particle's change in energy:

$$\Delta E_{\text{tot}} = W \tag{35}$$

We also know that the work done on the particle is equal to the dot product between the distance $\Delta\vec{x}$ the particle travels and F_{ext} the external force is applied to the particle.

$$W = \Delta\vec{x} \cdot \vec{F}_{\text{ext}} \tag{36}$$

With Eq.(35) and Eq.(36), we find that the total change in the particle's energy is

$$\Delta E_{\text{tot}} = \Delta\vec{x} \cdot \vec{F}_{\text{ext}} \tag{37}$$

Given that the power added to (or taken away from) a system is equal to the time derivative of the system's mechanical energy, we find that this power can be written as

$$P = \lim_{\Delta t \to 0} \frac{\Delta E_{\text{tot}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\vec{x} \cdot \vec{F}_{\text{ext}}}{\Delta t}, \tag{38}$$

[4] See supplement at the end of this section

which implies

$$P = \vec{F}_{\text{ext}} \cdot \vec{v} \tag{39}$$

where $\vec{v}$ is the velocity of the particle.