Lecture 06: Coupled Oscillations

In these notes we consider the dynamics of oscillating systems coupled together. To fully describe such systems we introduce the linear algebra concepts of eigenvectors and eigenvalues. We end by considering what the dynamics might look like if we considered an arbitrarily large system of oscillators together.

1 Oscillators are usually coupled

So far we've been studying oscillators individually. This is good for getting a handle on the techniques used to understand oscillations, but we eventually have to move beyond this simplification because in real-life oscillators rarely exist by themselves. Instead, real oscillators are almost always coupled to other systems and, in particular, other oscillators. For example, the chair or bed you're sitting on is composed of many atoms linked together in configurations which very much resemble chains of harmonic oscillators. We would eventually hope to describe something of the physics of such systems, so to do so we need to ask a new question.

Framing Question

How do we model many oscillators which are coupled together?

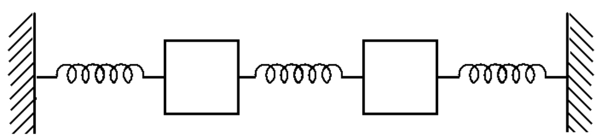


Figure 1: Simplest coupled oscillator system. Most oscillators in real life are coupled to other oscillators.

We will start simply first. In Fig. 1 we display two oscillators coupled end-to-end, a configuration we call **in series.** Our goal would be to characterize a system such as that in Fig. 1 (and its more complicated generalizations) as completely as we characterized the simpler one-oscillator systems.

2 Two Coupled Oscillators

We begin (as we usually do) with the simplest possible system. After we sufficiently understand this system, we will introduce additional features to bring us closer to the complexity which better resembles real systems. In Fig. 2, we reproduce our above coupled-oscillator picture with a specification of the relevant coordinates and parameters. We have two particles of the same mass m connected by a spring of spring constant k. Each mass is also connected to an adjacent wall by a spring of spring constant k. We define the positions of the left and right mass by x_1 and x_2, respectively, and we will take the stable equilibrium of the system to be defined at $x_1 = x_2 = 0$. Our objective is to find the equations defining the dynamics of this system and then find the general solution to those equations.

We start off with Newton's second law for both masses:

$$m\ddot{x}_1 = F_{\text{net},1} = -kx_1 + k(x_2 - x_1) \tag{1}$$

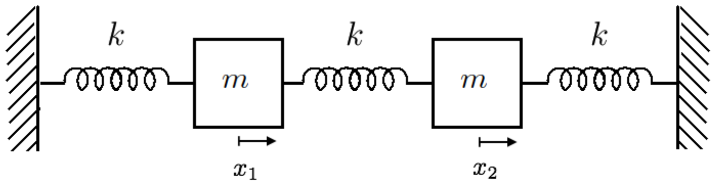


Figure 2: System of two coupled oscillators.

$$m\ddot{x}_2 = F_{\text{net},2} = -k(x_2 - x_1) - kx_2 \tag{2}$$

Each of the spring force terms in Eq.(1) and Eq.(2) can be determined by taking one of the masses to be stationary and considering what forces would be exerted on the other mass as it moves from its center. Now, we would like to solve this system of equations, but it appears that to solve the x_1 equation, we would need to know x_2, but to figure out x_2, we would need to know x_1. So this system may initially appear impossible to solve.

Progress, can be made through superposition. Let us add Eq.(1) and Eq.(2):

$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= m\ddot{x}_1 + m\ddot{x}_2 \\ &= -kx_1 + k(x_2 - x_1) - k(x_2 - x_1) - kx_2 \\ &= -k(x_1 + x_2) \end{aligned} \tag{3}$$

and then let us subtract Eq.(1) and Eq.(2):

$$\begin{aligned} m(\ddot{x}_1 - \ddot{x}_2) &= m\ddot{x}_1 - m\ddot{x}_2 \\ &= -kx_1 + k(x_2 - x_1) + k(x_2 - x_1) + kx_2 \\ &= -3k(x_1 - x_2) \end{aligned} \tag{4}$$

Considering both sides of Eq.(3) and both sides of Eq.(4), we find that our system is immediately soluble if we define

$$x_+ = x_1 + x_2 \quad \text{and} \quad x_- = x_1 - x_2. \tag{5}$$

With these definitions the above equations of motion become

$$\ddot{x}_+ = -\omega_0^2 x_+ \quad \text{and} \quad \ddot{x}_- = -3\omega_0^2 x_-, \tag{6}$$

where we defined the natural frequency $\omega_0 = \sqrt{k/m}$. We have thus managed to transform our system of two coupled differential equations in Eq.(1) and Eq.(2), into two independent simpler harmonic oscillator equations of motion. We well know how to find the general solution to the equations in Eq.(6) and so by Eq.(5) we can also find the general solution for x_1 and x_2. From our previous work, we have

$$x_+(t) = A_+ \cos(\omega_0 t - \phi_+) \quad \text{and} \quad x_-(t) = A_- \cos(\sqrt{3}\omega_0 t - \phi_-) \tag{7}$$

where $A_\pm$ and $\phi_\pm$ are arbitrary constants with units of meters and radians respectively. Inverting Eq.(5), we have

$$x_1 = \frac{1}{2}(x_+ + x_-) \quad \text{and} \quad x_2 = \frac{1}{2}(x_+ - x_-). \tag{8}$$

Thus, with Eq.(7), the general solution for Eq.(1) and Eq.(2) is

$$\begin{aligned} x_1(t) &= \frac{1}{2}A_+\cos(\omega_0 t - \phi_+) + \frac{1}{2}A_-\cos(\sqrt{3}\omega_0 t - \phi_-) \\ x_2(t) &= \frac{1}{2}A_+\cos(\omega_0 t - \phi_+) - \frac{1}{2}A_-\cos(\sqrt{3}\omega_0 t - \phi_-) \end{aligned} \tag{9}$$

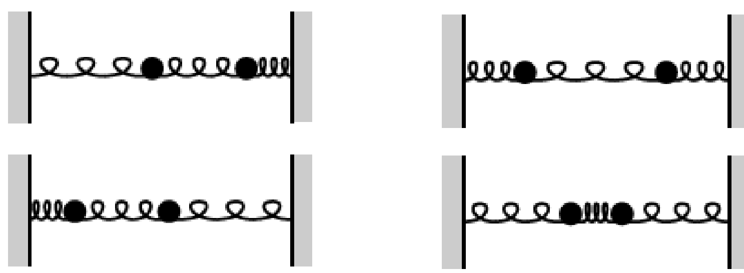
Since A_+ and A_- are undetermined constants we can in practice rescale them by 2 to absorb the factors of $1/2$ in Eq.(9). Also, the symmetry of the solution Eq.(9) suggests a convenient matrix representation. Letting x_1 and x_2 be the two components of a two-dimensional column vector, we can write Eq.(9) as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_+\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_0 t - \phi_+) + A_-\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\sqrt{3}\omega_0 t - \phi_-) \tag{10}$$

where we absorbed the factors of $1/2$ into a redefinition of $A_\pm$.

Eq.(10) turns out to be the most convenient form of the general solution to Eq.(1) and Eq.(2). It separates the two distinct sinusoidal contributions to the general solution and parameterizes each sinusoid by a vector. These vectors and sinusoids are important because together they make up all possible motions of the system. Indeed, *any* oscillating setup of the system depicted in Fig. **??** can be represented by Eq.(10) with appropriate choices of A_+ and A_-.

2.1 Normal Modes



(a) Symmetric motion (b) Antisymmetric motion

Figure 3: Normal mode motions for Fig. **??**. Figure from [1].

In Eq.(10), the vectors multiplying each sinusoid are called the **normal modes** of the motion:

$$\text{normal modes:} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{11}$$

and the angular frequencies at which the sinusoids oscillate are called the **normal mode angular frequencies** or simply (and somewhat incorrectly) the **normal mode frequencies** of the motion:

$$\text{normal mode frequencies:} \quad \omega_0 \quad \text{and} \quad \sqrt{3}\omega_0. \tag{12}$$

From Eq.(10) it should be clear that the normal mode $(1,1)$ oscillates with a frequency ω_0 and the normal mode $(1,-1)$ oscillates with a frequency $\sqrt{3}\omega_0$.

The vectors in Eq.(11) represent the building blocks of the coupled oscillator motion. We can understand what they physically represent by considering what Eq.(10) predicts about the motion in Fig. **??**. If we set A_- to zero in Eq.(10), we find that $x_1(t) = x_2(t)$ meaning that in Fig. **??** the two masses always move in the same direction with the same displacement while keeping the spring between them unstretched (See Fig. 3a). Such motion occurs with frequency ω_0. Alternatively if we set $A_+ = 0$ in Eq.(10), we find that $x_1 = -x_2$ meaning that in Fig. **??** the two masses always move in opposite directions with the same magnitude of displacement (See Fig. 3b). Such motion occurs with frequency $\sqrt{3}\omega_0$.

Since the general solution Eq.(10) consist of a linear combination of this $x_1 = x_2$ and $x_1 = -x_2$ motion, all the various possible motions of Fig. **??** can be reduced to a linear combination of x_1 and x_2 moving together and x_1 and x_2 moving with opposite displacements.

3 Two coupled oscillators - redux

For the simplest system imaginable (i.e., the one in Fig. **??**) we achieved our goal. We found the dynamical equations governing the system and solved them to find the most general description of motion. Along the way we introduced the concepts of normal modes and normal frequencies to help us better understand the resulting motion.

But all of this was largely due to luck (or fortuitous insight). We started with Eq.(1) and Eq.(2), and we were fortunate enough to find a trick which allowed us to reduce our coupled system of equations to two independent equations which could then be easily solved. But what would we do if we couldn't immediately see the best way to perform this reduction? For example, if all the spring constants and masses in Fig. **??** were different, how then would we solve the equation of motion?

It turns out there is a more reliable method of solution than the one we stumbled upon. The method begins by writing Eq.(1) as a matrix equation. First, dividing through by m and grouping terms associated with the same position variable, we have

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_1, \tag{13}$$

$$\ddot{x}_2 = \omega_0^2 x_1 - 2\omega_0^2 x_2. \tag{14}$$

Given the rules of matrix multiplication, we can write this system as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{15}$$

To solve Eq.(15) we employ that tried and true method of solving linear differential equations: Guess and check! Given our vector expression for the coordinates, and our prior knowledge of normal mode solutions, we guess the solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}, \tag{16}$$

where A, B, and α are undetermined constants. Now, our goal is to find the A, B, and α which make Eq.(16) a valid solution of Eq.(15). Inserting Eq.(16) into Eq.(15) we find,

$$\alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t} = \begin{pmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t} \tag{17}$$

or,

$$\begin{aligned} 0 &= \left[\begin{pmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 \end{pmatrix} - \alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} A \\ B \end{pmatrix} \\ &= \begin{pmatrix} -2\omega_0^2 - \alpha^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 - \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \end{aligned} \tag{18}$$

Now, our reframed goal is to find the A, B, and α which satisfy Eq.(18). Automatically we already know one (albeit trivial) solution. Even though it defines a completely stationary system, setting $A = B = 0$ certainly satisfies Eq.(18). Indeed this solution is what we would find if we were to invert the matrix in Eq.(18). That is we could define

$$\hat{\Omega} - \alpha^2 \mathbb{I} \equiv \begin{pmatrix} -2\omega_0^2 - \alpha^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 - \alpha^2 \end{pmatrix}, \tag{19}$$

where $\mathbb{I}$ is the 2×2 identity matrix, and then, *presuming this matrix was invertible*, we could then find the solution to Eq.(18) through

$$\begin{aligned} 0 &= \left(\hat{\Omega} - \alpha^2 \mathbb{I}\right) \begin{pmatrix} A \\ B \end{pmatrix} \\ \left(\hat{\Omega} - \alpha^2 \mathbb{I}\right)^{-1} 0 &= \left(\hat{\Omega} - \alpha^2 \mathbb{I}\right)^{-1} \left(\hat{\Omega} - \alpha^2 \mathbb{I}\right) \begin{pmatrix} A \\ B \end{pmatrix} \\ 0 &= \begin{pmatrix} A \\ B \end{pmatrix}. \end{aligned} \tag{20}$$

Thus, whenever we can invert Eq.(19), we will always find the trivial solution Eq.(20). We do not want this. Rather we want non-trivial solutions where A and/or B are nonzero. To find these solutions we will have to assume that we *cannot* invert Eq.(19) and hence cannot perform the calculation leading to Eq.(20). From linear algebra, to say that we cannot invert a matrix is to say that the inverse does not exist. For a matrix $\hat{A}$ the inverse is defined schematically as

$$\hat{A}^{-1} = \frac{1}{\det \hat{A}} \times \Big(\text{"some other matrix related to } \hat{A}\text{"}\Big), \tag{21}$$

where $\det \hat{A}$ is the determinant of $\hat{A}$. The "some other matrix..." part is not important for us. What is important is the $1/\det \hat{A}$ part. Given that we cannot divide a matrix by 0, the inverse $\hat{A}^{-1}$ does not exist whenever $\det \hat{A}$ does not exist.

Existence of an Inverse: The inverse of the matrix $\hat{A}$ exists if and only if the determinant of $\hat{A}$ is non-zero.

Therefore, in order to find the nontrivial solutions to Eq.(18), we need to consider the case where Eq.(19) is not invertible, namely the case where its determinant is zero. Computing the determinant of Eq.(19) and setting the result to zero, we find

$$0 = \begin{vmatrix} -2\omega_0^2 - \alpha^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 - \alpha^2 \end{vmatrix} = \alpha^4 + 4\omega_0^2\alpha^2 + 3\omega_0^4 \tag{22}$$

Eq.(22) can be solved for α^2 (and in turn α) through the quadratic equation. Doing so yields

$$\alpha_\pm^2 = -2\omega_0^2 \pm \frac{1}{2}\sqrt{16\omega_0^4 - 12\omega_0^4} = -2\omega_0^2 \pm \omega_0^2. \tag{23}$$

We thus find the solutions

$$\alpha_+^2 = -\omega_0^2 \to \begin{cases} \alpha_+ = i\omega_0 \\ \alpha_+ = -i\omega_0 \end{cases} \quad \text{and} \quad \alpha_-^2 = -3\omega_0^2 \to \begin{cases} \alpha_- = i\sqrt{3}\omega_0 \\ \alpha_- = -i\sqrt{3}\omega_0. \end{cases} \tag{24}$$

Now, it is time to find the coefficients A and B. Given the α^2 values we found above, we determined based on a "non-invertible" condition, we should expect to find non-trivial (i.e., nonzero) values for A and B. We will determine these values from Eq.(24) and Eq.(18). Given the α_+^2 solutions, Eq.(18) states that the associated coefficients A_+ and B_+ must satisfy

$$0 = \begin{pmatrix} -2\omega_0^2 - \alpha_+^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 - \alpha_+^2 \end{pmatrix} \begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \begin{pmatrix} -\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} A_+ \\ B_+ \end{pmatrix}. \tag{25}$$

The most general (A_+, B_+) vector which satisfies this equation is of the form

$$\begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{26}$$

where A_+ is an arbitrary constant. This (A, B) solution is associated with both $\alpha_+ = i\omega_0$ and $\alpha_+ = -i\omega_0$. Similarly, given the the α_-^2 solutions, Eq.(18) states that the associated coefficients A_- and B_- must satisfy

$$0 = \begin{pmatrix} \omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2 \end{pmatrix} \begin{pmatrix} A_- \\ B_- \end{pmatrix}. \tag{27}$$

The most general (A_-, B_-) vector which satisfies this equation is of the form

$$\begin{pmatrix} A_- \\ B_- \end{pmatrix} = A_- \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{28}$$

where A_- is an arbitrary constant. This (A, B) solution is associated with both $\alpha_- = i\sqrt{3}\omega_0$ and $\alpha_- = -i\sqrt{3}\omega_0$.

At this point we are basically done: Our goal was to find the A, B, and α which allowed Eq.(16) to be a solution of Eq.(15). We found a set of four such values:

$$A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t}, \quad B_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t}, \quad A_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t}, \quad \text{and} \quad B_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t}, \tag{29}$$

where $A_\pm$ and $B_\pm$ are our four arbitrary constants set by initial conditions. Given what we know about solutions to linear differential equations, it should be clear that the most general solution to Eq.(15) is a linear combination of the solutions in Eq.(29). That is the most general solution is

$$\begin{aligned} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\omega_0 t} + B_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_0 t} \\ + A_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\omega_0 t} + B_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\omega_0 t}, \end{aligned} \tag{30}$$

Given that x_1 and x_2 must be real, we have to take the real part of Eq.(30) in order to find the physical solution. In finding this solution we take $A_\pm$ and $B_\pm$ to be complex, and we use the identity

$$\text{Re}\left[Ae^{i\theta} + Be^{-i\theta}\right] = \text{Re}[A + B]\cos\theta - \text{Im}[A - B]\sin\theta = C\cos(\theta - \phi), \tag{31}$$

where C and ϕ are real quantities defined in terms of Re$[A + B]$ and Im$[A - B]$. Taking the real part of Eq.(30) and using Eq.(31), we thus find the physical part of the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_+) + C_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\sqrt{3}\omega_0 t - \phi_-) \tag{32}$$

which is identical to the solution Eq.(10) we found previously.

3.1 Discussion: Eigenvalues and Eigenvectors

Now I can probably guess what you're thinking: We went through all that mathematics to derive a result we obtained much more simply before. What was the point of all this?

The first point has to do with generality. In the method we initially used to derive Eq.(10), we took advantage of the simplicity of our system to to find a way to separate the equations of motion into two simpler forms. In more complex system with less symmetry amongst the defining parameters, it would be more difficult to find such convenient linear combinations. In fact the algorithmic way to find combinations analogous to Eq.(5) in more complex systems is to do precisely what we did in the second method.

The second point is pedagogical. By solving Eq.(15) for the solution Eq.(16) we tacitly made use of an important construct in linear algebra. Dropping the exponential factor on either side of Eq.(17) and defining the matrix as $\hat{\Omega}$ and the vector (A, B) as $\vec{x}_0$, we have the equation

$$\hat{\Omega}\,\vec{x}_0 = \alpha^2\vec{x}_0. \tag{33}$$

In Eq.(33), we have a matrix $\hat{\Omega}$ multiplying a vector $\vec{x}_0$ and producing a scalar α^2 multiplying the vector. Whenever we have a situation where matrices, vectors, and scalars are related as they are in Eq.(33), we term the scalar α^2 the **eigenvalue** of the matrix $\hat{\Omega}$ and the vector $\vec{x}_0$ the **eigenvector** of the matrix. In solving Eq.(16) for (A, B) and α we were in essence solving the eigenvalue problem of $\hat{\Omega}$. The method we used to find this solution (namely, setting $\det(\hat{\Omega} - \alpha^2\mathbb{I})$ to zero and solving for α^2 and $\vec{x}_0$ which satisfied Eq.(33)) is the standard one used to find the eigenvalues and eigenvectors of a matrix. If you continue to study science or engineering, you will almost certainly see this procedure again, so it was worth introducing now.

Normal Modes and Eigenvectors: We should note that the normal modes we previously found (Eq.(11)) are the eigenvectors of the matrix $\hat{\Omega}$ and the normal mode frequencies (Eq.(12)) are the imaginary parts of the square roots of the eigenvalues. This is general for any coupled oscillator system we consider. The eigenvalues and eigenvectors of the interaction matrix (i.e., whatever matrix that takes the place of $\hat{\Omega}$ in this problem) are related to the normal mode frequencies and normal modes, respectively, of the oscillating system: Given the equation of motion

$$\ddot{\vec{x}}(t) = \hat{\Omega}\,\vec{x}(t), \tag{34}$$

were $\vec{x} = (x_1, x_2, \ldots, x_N)$ and $\hat{\Omega}$ is an $N \times N$ matrix, we have the eigenvalue-eigenvector equation

$$\hat{\Omega}\,\vec{x}_0 = \alpha^2\vec{x}_0 \tag{35}$$

where possible values of $\vec{x}_0$ define the normal modes and possible values of $\mathrm{Im}(\alpha)$ define the normal mode frequencies of the system.

4 $N = 3$ oscillators

Having considered a system of two coupled masses, any slightly more complex generalizations might involve longer calculations but would use the same methods we have already developed. Take the system of three oscillating masses depicted above. Expressing the equations of motion as a matrix equation, we would find

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{36}$$

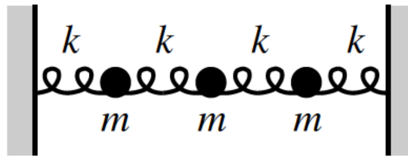


Figure 4: Three oscillators coupled in series. Figure from [1]

where $\omega_0^2 = k/m$. The key to finding the general solution to this equation of motion would be to first find the eigenvectors and eigenvalues of the 3×3 matrix. In analogy to the system of two coupled oscillators considered in the previous section, we would find three normal modes and three normal mode frequencies making up the most general solution. The solution would look something like

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = C_1\,\vec{v}_1 \cos(\omega_1 t - \phi_1) + C_2\,\vec{v}_2 \cos(\omega_2 t - \phi_2) + C_3\,\vec{v}_3 \cos(\omega_3 t - \phi_3), \tag{37}$$

where ω_1, ω_2, ω_3 and $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are the normal mode frequencies and normal modes, respectively, of Fig. 4. Finding these quantities would require us to compute a determinant similar to Eq.(22) and then solve for the general coefficients of the vector for each eigenvalue we find.

5 $N \to \infty$ oscillators

All of this is straight forward enough for the system in Fig. 4, but what would we do if we had ten oscillators, or twenty? For such large N systems, we could no longer work analytically and we would instead need computational algorithms to find the normal mode and normal mode frequencies.

But interestingly, if we take N to be even larger than 10 or 20, in particular if we take $N \to \infty$ the translational symmetry of the resulting problem would allow us to return to analytic methods. Consider the figure below consisting of N identical masses coupled through identical springs.

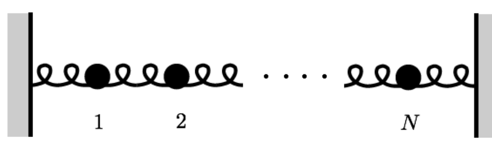


Figure 5: N oscillators coupled in series. What happens to the equation of motion for $N \to \infty$? Figure from [1]

The equation of motion for the mass j within this coupled series would be

$$m\ddot{x}_j = k(x_{j+1} - x_j) - k(x_j - x_{j-1}) \qquad \text{[For } 1 \le j \le N\text{]} \tag{38}$$

where $x_{N+1} = x_{-1} = 0$. How could we solve this equation? We could try to create a large $N \times N$ matrix analogous to that in Eq.(36), but there is in fact a simpler method. The simpler method makes use of our old friend from calculus: **the continuum limit.** To take Eq.(38) to the continuum limit, we first define a rest-length lattice spacing of size a between the oscillators in Fig. 5. We then take Na to define the fixed length L,

$$Na = L \tag{39}$$

and consider the limit of Eq.(38) as $N \to \infty$. This limit allows us to promote the index-valued positions $x_j(t)$ to functions $x(s,t)$. With this promotion Eq.(38) becomes

$$m\ddot{x}(s) = k\left(x(s+a) - x(s)\right) - k\left(x(s) - x(s-a)\right). \tag{40}$$

We then define a fixed mass density λ as

$$\lambda = \frac{m}{a}, \tag{41}$$

and a tension force E as

$$E = ka. \tag{42}$$

We impose that both λ and E are independent of a and remain the same regardless of how small our lattice spacing becomes. This imposition would in turn require m and k to have some dependence on a. This stipulation allows us to complete our continuum limit. Dividing both sides of Eq.(40) by a and taking $a \to 0$ we obtain

$$\begin{aligned}
\lim_{a\to 0} \frac{m}{a}\ddot{x}(s) &= \lim_{a\to 0} \frac{k}{a}\left(x(s+a) - x(s)\right) - k\left(x(s) - x(s-a)\right) \\
\lambda\ddot{x}(s) &= E \lim \frac{1}{a^2}\left[\left(x(s+a) - x(s)\right) - \left(x(s) - x(s-a)\right)\right] \\
&= E \lim \frac{1}{a}\left[\frac{\left(x(s+a) - x(s)\right)}{a} - \frac{\left(x(s) - x(s-a)\right)}{a}\right] \\
&= Ex''(s),
\end{aligned} \tag{43}$$

where $\dot{()}$ refers to a time derivative and $()'$ refers to an s derivative. Now, the function $x(s)$ is actually also a function of time, so it proves more appropriate to write this variable dependence and the corresponding derivatives explicitly. Doing so we find

$$\lambda \frac{\partial^2}{\partial t^2} x(s,t) = E \frac{\partial^2}{\partial s^2} x(s,t), \tag{44}$$

the dynamical equation governing Fig. 5 when $N \to \infty$. In most contexts, we simply call Eq.(44) **the wave equation.**

References

[1] D. Morin, *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.