**Lecture 08: Traveling Waves and Boundary Interactions**

In these notes, we solve the wave equation for traveling wave solutions and calculate the transmission and reflection coefficients characterizing how waves propagate across boundaries. We also discuss waves traveling through media and the energy dissipation that can result. We end with a short discussion of sound waves.

# 1 Waves *en route*

In the previous notes, we successfully derived the wave equation describing how strings underwent both longitudinal and transverse[1] oscillations. For transverse oscillations defined by a vertical displacement $y(x,t)$, we found the wave equation was given by

$$\frac{\partial^2}{\partial t^2} y(x,t) = v^2 \frac{\partial^2}{\partial x^2} y(x,t), \tag{1}$$

where $v$ (which is $\sqrt{T/\mu}$ for the string) was a quantity with units of velocity that we had yet to interpret. In deriving this equation, we considered the string to reside within a specific domain of space and considered the wave motion propagating within this domain. Such an assumption was a convenient starting point in analyzing waves, but not all waves exist bounded in fixed domains. For example, light and sound waves (both of which are described by wave equations) travel from one place to another interacting with the various media they come in contact with.

With regard to light, we are able to see because light waves are reflected off various objects in our environment and travel to the rod and cone cells in our eyes. During travel, the intensity of the light can decrease if it has to propagate through fog, or it might vanish entirely if it was incident on an opaque object.

With regard to sound, if you can hear cars driving outside or your neighbors sitting rooms away, it is because sound waves are traveling through the air, being transmitted through walls and boundaries separating and finally hitting your ear drums. Not all of the sound waves emitted by their respective sources reach you. Some of these waves are reflected back toward their source and others decay away quickly.

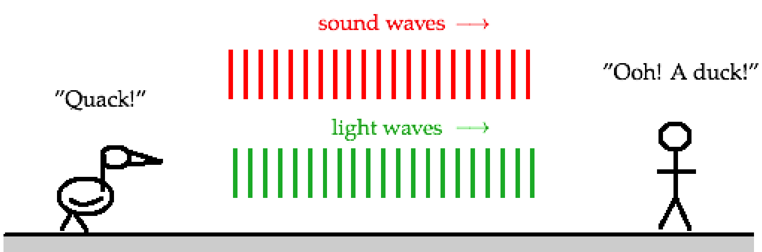


Figure 1: We can hear and see objects around us due to traveling sound and light waves. How do we mathematically describe such waves?

We would like to be able to describe such propagation. Since we have only so far considered bounded

[1] As a useful mnemonic to differentiate the two, remember that longitudinal waves travel "along" the axis of propagation

waves, we would need to expand our conception of solutions to the wave equation in order to describe traveling waves. To this end, the framing question for this lesson is

**Framing Question**

How do we model the ways waves travel through space and are reflected, transmitted, and dispersed through boundaries?

# 2 Traveling wave solutions

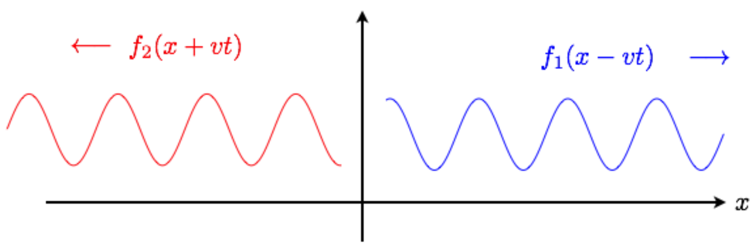


Figure 2: The two terms in the general solution Eq.(4) define wave propagation at the same speed but in different directions.

Our goal is to solve Eq.(1) such that our solutions describe traveling waves. Our method of solution will parallel our previous solution methods (namely guess and check), but before we make the attempt, let us discuss some properties we want these traveling wave solutions to have.

Unlike our standing wave solutions (which were defined only within the domain $[0, L]$) we want our traveling waves to be defined for a much larger space of $x$ values. For simplicity, we will take this larger space of $x$ values to be the entire real line[2], i.e., we want our wave to be defined for $x = +\infty$ and $x = -\infty$.

Also, we will take our waves to be propagating with a finite velocity $v$. Concerning $v$, we would want our waves to propagate such that if we translate them by a distance $\Delta x$ in the opposite direction of the propagating wave front, then this would be tantamount to not moving at all and instead waiting a time $\Delta t = \Delta x/v$ for the wave to advance by the position $\Delta x$.

To satisfy this property, our wave $y(x,t)$ must be a function of $x - vt$. We can see this by noting that if we take $x \to x - \Delta x$ (i.e., move by $\Delta x$ in the opposite direction of $+v$), then the function $f_1(x - vt)$ becomes

$$f_1(x - \Delta x - vt) = f_1(x - v(t + \Delta t)) \tag{2}$$

which is equivalent to taking $t \to t + \Delta t$ if $\Delta t = \Delta x/v$. Therefore, as our guess for a solution to Eq.(1), we take $y(x,t) = f_1(x - vt)$ where $f_1$ is any sufficiently well-behaved[3] function. Checking this solution we have

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} y(x,t) &= v^2 \frac{\partial^2}{\partial x^2} y(x,t) \\
\frac{\partial^2}{\partial t^2} f_1(x - vt) &= v^2 \frac{\partial^2}{\partial x^2} f_1(x - vt) \\
(-v)^2 f_1(x - vt) &= v^2 f_1(x - vt)
\end{aligned}$$

[2] In reality waves always interact with *something* (either the source of the wave or its target) at some point during their propagation so their domain is actually constricted.

[3] By "well-behaved" we mean there are no singularities.

$$f_1(x - vt) = f_1(x - vt). \tag{3}$$

Thus *any* function which is a function of the quantity $x - vt$ is a solution to the wave equation. Through a similar calculation, we can show that the solution $y = f_2(x + vt)$ (where $f_2$ is not necessarily equivalent to $f_1$) is also a solution to Eq.(1). These two solutions $f_1(x - vt)$ and $f_2(x + vt)$ represent wave propagation in two different directions: The solution $f_2(x + vt)$ represents a wave propagating to the left, as opposed to the rightward propagation of $f_1(x - vt)$. Now, given that the general solution to a linear differential equation is a linear combination of the possible solutions, we find that the general solution to Eq.(1) is

$$y(x, t) = f_1(x - vt) + f_2(x + vt) \tag{4}$$

where the coefficients of the linear combination were absorbed into redefinitions of $f_1$ and $f_2$. Although, it does not look like it, Eq.(4) includes the standing wave solutions we found through separation of variables in the previous notes. Thus, Eq.(4) is indeed the most general (although not always the most useful) form of the solutions to Eq.(1).

## 2.1 The Connection between Standing and Traveling Waves

We stated that Eq.(4) is the true general solution for a wave equation. What relationship does it have with the previous solution we found for standing waves?

To understand the relationship, we consider Eq.(4) with two sinusoidal waves of equal phase and amplitude but traveling in opposite directions. We define this wave amplitude as $u(x, t)$:

$$u(x, t) = \frac{1}{2}\left[A\cos\left(k(x - vt)\right) - A\cos\left(k(x + vt)\right)\right], \tag{5}$$

where $k$ is an as of yet unspecified wave number. Using the sum of angles formula for cosine functions, we find that Eq.(5) becomes

$$u(x, t) = A\sin(kx)\sin(\omega t), \tag{6}$$

where we used the definition of angular frequency to replace $kv$ with $\omega$. From here, er vsn extend this result by presuming that instead of having two waves traveling on the string, we have infinitely many waves all of the form Eq.(5) except each wave's wavenumber $k$ depends on an index $n = 1, 2, \ldots$ such that $k_n = n\pi/L$ (where $L$ is some length of interest). The coefficient $A$ is then reasonably expected to depend on $n$, so we would replace $A$ with $A_n$. The total wave displacement would then be

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} u_n(x, t) \\ &= \sum_{n=1}^{\infty} \frac{A_n}{2}\left[\cos\left(k_n(x - vt)\right) - \cos\left(k_n(x + vt)\right)\right] \\ &= \sum_{n=1}^{\infty} A\sin(k_n x)\sin(\omega_n t), \end{aligned} \tag{7}$$

which is the solution we found for the waves on a string bounded within $x = 0$ and $x = L$. Eq.(7) is somewhat less general than the Fourier series solution found before in that it satisfies $y(x, 0) = 0$, and thus an initial condition has (implicitly) already been imposed. To find the more general solution, we would have needed to begin with a more general combination of sines and cosines in Eq.(5). In either case, we see that standing waves can be represented by linear combinations of traveling waves moving in opposite directions.

## 2.2 Fourier Integrals and Periodic Motion

Given that our focus is on periodic motion, it is natural to first explore sinusoidal solutions in Eq.(4). So we will take our function $f_1$ and $f_2$ to be a linear combination of sinusoids:

Thus the solution Eq.(4) becomes

$$y(x,t) = A\cos\left[k(x-vt)\right] + B\sin\left[k(x-vt)\right] + C\cos\left[k(x+vt)\right] + D\sin\left[k(x+vt)\right] \tag{8}$$

where $A$, $B$, $C$, and $D$ are real quantities. For the purposes of calculation, it will prove easier to deal with a more general form of these sinusoidal solutions. We define the complex variable $z(x,t)$ such that $y(x,t)$ is the real part of $z(x,t)$:

$$y(x,t) = \text{Re}\left[z(x,t)\right], \tag{9}$$

We will take $z(x,t)$ to satisfy the same wave equation as $y(x,t)$,

$$\frac{\partial^2}{\partial t^2} z(x,t) = v^2 \frac{\partial^2}{\partial x^2} z(x,t), \tag{10}$$

with the same form of the general solution

$$z(x,t) = \widetilde{f}_1(x-vt) + \widetilde{f}_2(x+vt) \tag{11}$$

The fact that $z(x,t)$ is complex allows us to consider sinusoidal solutions written as complex exponentials. Namely, the complex analog of Eq.(8) is

$$z(x,t) = A_+ e^{ik(x-vt)} + B_+ e^{-ik(x-vt)} + A_- e^{ik(x+vt)} + B_- e^{-ik(x+vt)}, \tag{12}$$

where $A_\pm$ and $B_\pm$ are complex quantities. We include both $e^{iku}$ and $e^{-iku}$ type of functions in Eq.(12) because complex exponentials with opposite arguments are independent of one another. We can obtain Eq.(8) by applying Eq.(9) and taking Re$[A_+] = A$, Im$[A_+] = -B$, and so on. The way we will use Eq.(12), is that we will take its real part before we physically interpret any result computed from it. Recalling our previous nomenclature, for the forward moving wave solution $|A_+|$ is the amplitude of the wave, $k$ is the wave number, and $kv = \omega$ is the frequency.

In the previous lesson, we noted that solutions to the wave equation defined by a single wave number $k$ only represent one solution to the wave equation. For example, we found that one solution to the wave equation in a bounded domain was

$$A_n \sin\left(k_n vt\right) \sin\left(k_n x\right). \tag{13}$$

In order to find the general solution, we needed to sum this solution over all possible values of $k_n$. Given that each solution is associated with a specific coefficient, the general solution was then

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(k_n vt\right) \sin\left(k_n x\right) \tag{14}$$

Similarly, to find the most general form of Eq.(12) we need to sum over all possible values of $k$. Here, $k$ is a continuous (rather than discrete) variable so this summation will take the form of an integral. Moreover, the continuous analog of the $A_n$ in Eq.(14) is $A_+(k)$ and $A_-(k)$. Thus, the most general form of the sinusoidal solution Eq.(12) is

$$z(x,t) = \int_{-\infty}^{\infty} dk\, \left[A_+(k) e^{ik(x-vt)} + A_-(k) e^{ik(x+vt)}\right], \tag{15}$$

We do not need to include the $B$ coefficients from Eq.(12); since our integral is running from $-\infty$ to $+\infty$, the $e^{-ik(x-vt)}$ and $e^{-ik(x+vt)}$ solutions are automatically included in the negative domain of $k$. In the subsequent analysis, we will mostly be considering solutions of the form Eq.(11). However, Eq.(15) is important because it is related to field of mathematics called **Fourier analysis** which is built around the following theorem.

**Fourier's Theorem:** The two forms of the general solution to Eq.(10) (i.e., Eq.(11) and Eq.(15)) are actually equivalent. By a mathematical result called **Fourier's Theorem**, a general function $f(x)$ can be represented as

$$f(x) = \int_{-\infty}^{\infty} dk\, \phi(k) e^{ikx}, \tag{16}$$

where the function $\phi(k)$ is in turn given by

$$\phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx}. \tag{17}$$

Using Fourier's theorem we can express the arbitrary functions $\tilde{f}_1$ and $\tilde{f}_2$ used in Eq.(11) as

$$\tilde{f}_1(s) = \int_{-\infty}^{\infty} dk\, A_+(k) e^{iks} \quad \text{and} \quad \tilde{f}_2(s) = \int_{-\infty}^{\infty} dk\, A_+(k) e^{iks}, \tag{18}$$

where $A_+$ and $A_-$ are defined by equations similar to Eq.(17). With Eq.(16), we thus find

$$z(x,t) = \tilde{f}_1(x - vt) + \tilde{f}_2(x + vt) = \int_{-\infty}^{\infty} dk\, \left[A_+(k) e^{ik(x-vt)} + A_-(k) e^{ik(x+vt)}\right], \tag{19}$$

which establishes the equivalence between Eq.(11) and Eq.(15).

# 3 Waves changing media

With Eq.(15), we have at last found a general mathematical description of waves traveling in free space, but as mentioned in the introduction, waves rarely travel unimpeded. Rather, waves often interact with their surrounding by changing media or reflecting off of surfaces. For example, mirrors work by reflecting light directly back at its source, and we can see through to the bottom of pools because light is propagating from the bottom of the pool through the water and then through the air to our eyes. Whenever the medium through which a wave is propagating changes, its properties change. In this section, we study these changes using a simple model of a propagating string.

## 3.1 Reflection and Transmission

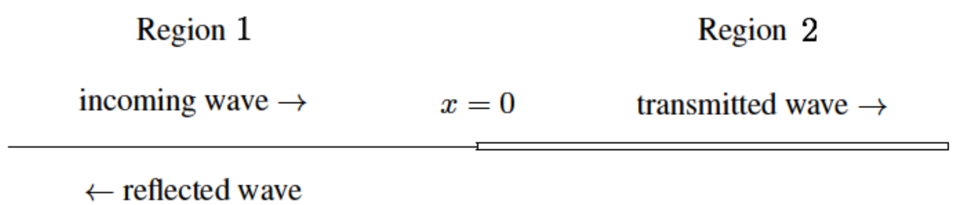


Figure 3: Two strings of different mass densities joined at $x = 0$. Figure from [1]

For the string system that we have been considering the "medium" through which the wave is propagating is the string itself. The defining quality of the string which defines the properties of propagation is its mass density $\mu$. Thus to study how the properties of waves change as they cross media, we have to consider how waves change as they transition between strings with different mass densities[4].

We begin with the system shown in Fig. 3 where a string of mass density $\mu_1$ for $x < 0$ is connected at the point $x = 0$ to a string of mass density $\mu_2$ for $x > 0$. We assume that for all time, we have a continuos wave $f_0(x - vt)$ coming in from the left (and we know it is coming in from the left because $f_0$ is a function of $x - vt$ and not $x + vt$). We want to know what happens to the wave after it reaches the origin.

We presume two things can happen: part of the wave will be transmitted to the string of mass density $\mu_2$ and part of the wave will be reflected back at its source. Namely, if we were to write the equation representing the total wave for $x < 0$ and $x > 0$ we would have

$$y(x,t) = \begin{cases} f_0(x - v_1 t) + f_R(x + v_1 t) & \text{for } x < 0 \\ f_T(x - v_2 t) & \text{for } x > 0 \end{cases} \tag{20}$$

where $R$ stands for the reflected wave and $T$ stands for transmitted wave. We use different velocities for $x < 0$ and $x > 0$ because the velocity of a wave on a string is defined by $v = \sqrt{T/\mu}$ and the sections of the string on either side of $x = 0$ have different densities. We note that the transmitted wave is traveling to the right because it is a function of $x - vt$, but the reflected wave is traveling to the left because it is a function of $x + vt$.

### 3.2 Continuity and Zero Net-Transverse Force

We will determine a relationship between $f_0$, $f_R$, and $f_T$ by using physical properties of the string. First, the string is continuous, so the total wave amplitude at $x = 0$ must be the same on both sides of the point connecting the two parts of the string. This means we must have

$$y(0^+, t) = y(0^-, t), \qquad \text{[Continuity condition]} \tag{21}$$

where $0^+$ defines $x = 0$ found by approaching the point from the right and $0^-$ defines $x = 0$ found by approaching the point from the left. In terms of our general solution for , this continuity condition becomes

$$f_0(-v_1 t) + f_R(v_1 t) = f_T(-v_2 t). \tag{22}$$

Next, any point on the string is essentially massless, so the total force exerted on that point must be zero. This must be true of the point joining the $x < 0$ and $x > 0$ regions of the string as well. For the string with transverse oscillations, the force arises from the tension $T$ and the slope of the string. We can define this slope by the angle $\theta$ the string makes with the horizontal. We will assume $\theta$ is small which is true for low amplitude waves. When a section of string is at an angle $\theta$ with the horizontal, the force which is being exerted on it to produce this angle has horizontal and vertical components. For a string of tension $T$, the horizontal component of the force at the point $x = 0$ coming from left part of the string is

$$F_x(x = 0) = -T\cos\theta \simeq -T. \tag{23}$$

We use a negative sign because the tension from the left part of the string is pulling the point at the origin toward the $x < 0$ direction. Similarly, the vertical component of the force coming from the left part of the string is

$$F_y(x = 0) = -T\sin\theta \simeq T\theta \simeq T\tan\theta = -T\left(\frac{\partial y}{\partial x}\right), \tag{24}$$

[4] More generally, we could also consider different tensions, but for simplicity we will take the tensions to be equal on both sides of the string.

where we used the small-angle approximation of $\sin\theta$ and $\tan\theta$. Here, the negative sign arises because if the slope of the string at the origin is positive then the left part of the string must be pulling down on the point at the origin. The requirement that the net-force on the part of the string at $x = 0$ is zero, amounts to requiring that the sum of **F** for $x < 0$ and **F** for $x > 0$ at $x = 0$ is zero. Given Eq.(23), this condition is automatically satisfied for the horizontal forces acting within the string. Namely we have

$$F_{\text{net},x}(x=0) = \lim_{x\to 0^+} F_x + \lim_{x\to 0^-} F_x \simeq -T + T = 0. \tag{25}$$

where we use the notation $x \to 0^+$ or $x \to 0^-$ to signify approaching $x = 0$ from the right or left, respectively. Conversely, we also require that the net-force in the $y$-direction is zero, and so we have

$$F_{\text{net},y}(x=0) = \lim_{x\to 0^+} F_y + \lim_{x\to 0^-} F_y \simeq -\lim_{x\to 0^+} T\left(\frac{\partial y}{\partial x}\right) + \lim_{x\to 0^-} T\left(\frac{\partial y}{\partial x}\right) = 0. \tag{26}$$

Therefore, for our string of constant tension, this condition of zero net-force in turn implies

$$T\left(\frac{\partial y}{\partial x}\right)_{0^+} = T\left(\frac{\partial y}{\partial x}\right)_{0^-}. \qquad \text{[Zero net-transverse force condition]} \tag{27}$$

Given that the string tension is the same on both sides of the boundary, written in terms of the general solution Eq.(3.1), this result becomes

$$f_0'(-v_1 t) + f_R'(v_1 t) = f_T'(-v_2 t) \tag{28}$$

In summary, the two conditions that the string displacement at an interface (placed at $x = 0$) must satisfy are

1. **Continuity:** The string must be continuous across the interface (Eq.(21)). This means the displacement $y(x,t)$ when approaching the interface from the left must be the same as that when approaching the interface from the right.

2. **Zero net-transverse force:** The net-vertical force at the interface must be zero (Eq.(27)). This is to ensure there is no net-force applied to the infinitesimal mass at the interface; such a net force would yield an infinite acceleration.

### 3.3 Reflection and Transmission Coefficients

Eq.(22) and Eq.(28) are convenient starting points for deriving properties relating the incident wavefront $f_0$ to the reflected and transmitted wavefronts $f_R$ and $f_T$. We can go even further by positing a standard form for these waves. For simplicity, we will return to the complex exponential form of the general sinusoidal solution [5]. Our traveling wave solutions are then

$$\widetilde{f}_0(u_+) = A_0 e^{ik_1 u_+} + B_0 e^{-ik_1 u_+}, \qquad \widetilde{f}_R(u_-) = A_R e^{ik_1 u_-} + B_R e^{-ik_1 u_-}, \qquad \widetilde{f}_T(u_+) = A_T e^{ik_2 u_+} + A_T e^{-ik_2 u_+}, \tag{29}$$

where $u_+ = x - vt$ and $u_- = x + vt$ and where we use $k_1$ and $k_2$ to refer to the wave numbers of the $\mu_1$ string and $\mu_2$ string, respectively. Given the complex exponential analog of Eq.(29), we find that the total wave for $x < 0$ is

$$z(x,t) = A_0 e^{ik_1(x-v_1 t)} + B_0 e^{-ik_1(x-v_1 t)} + A_R e^{ik_1(x+v_1 t)} + B_R e^{ik_1(x+v_1 t)} \qquad \text{[For } x<0\text{]} \tag{30}$$

[5]To be more precise we should have performed this analysis with the function $z(x,t)$, but this measure of sloppiness does not affect the final results. If we find any complex quantities in a final result, we will simply take the real part and assume we implemented the procedure correctly from the beginning.

and the total wave for $x > 0$ is

$$z(x,t) = A_T e^{ik_2(x+v_2 t)} + B_T e^{ik_2(x+v_2 t)} \qquad \text{[For } x > 0\text{]}. \tag{31}$$

We note that because string 1 and 2 differ in their density, the parameters in the system which are dependent on length scale are different for the two sections of the strings. Namely, any quantity which includes units of distance is different when it is defined on the leftside versus on the rightside of Fig. 3. However, any quantity with out any units of distance is the same across the string. This means that although $k$ and $v$ (with units of $\mathrm{m}^{-1}$ and m/s, respectively) are not the same across the two sections of the string, $\omega = kv$ (with units of rad/sec) is the same. Therefore, Eq.(30) and Eq.(31) becomes

$$z(x,t) = A_0 e^{i(k_1 x - \omega t)} + B_0 e^{-i(k_1 x - \omega t)} + A_R e^{i(k_1 x + \omega t)} + B_R e^{-i(k_1 x + \omega t)} \qquad \text{[For } x < 0\text{]} \tag{32}$$

or

$$z(x,t) = A_T e^{i(k_2 x - \omega t)} + B_T e^{-i(k_2 x - \omega t)} \qquad \text{[For } x > 0\text{]}. \tag{33}$$

Now, imposing Eq.(21) on these forms of $z(x,t)$, we find

$$\begin{aligned} A_T e^{-i\omega t} + B_T e^{i\omega t} &= A_0 e^{-i\omega t} + B_0 e^{i\omega t} + A_R e^{i\omega t} + B_R e^{-i\omega t} \\ &= (A_0 + B_R) e^{-i\omega t} + (B_0 + A_R) e^{i\omega t}, \end{aligned} \tag{34}$$

which given the linear independence of $e^{i\omega t}$ and $e^{-i\omega t}$, leads to the two equations

$$A_T = A_0 + B_R, \qquad B_T = B_0 + A_R. \tag{35}$$

Similarly, imposing Eq.(27) on these forms, gives us

$$\begin{aligned} ik_2 A_T e^{-i\omega t} - ik_2 B_T e^{-i\omega t} &= ik_1 A_0 e^{-i\omega t} - ik_1 B_0 e^{i\omega t} + ik_1 A_R e^{i\omega t} - ik_1 B_R e^{-i\omega t} \\ 0 &= (k_1 A_0 - k_1 B_R) e^{i\omega t} + (-k_1 B_0 + k_1 A_R) e^{i\omega t}, \end{aligned} \tag{36}$$

which then yields the other two equations

$$k_2 A_T = k_1 (A_0 - B_R), \qquad -k_2 B_T = k_1(-B_0 + A_R). \tag{37}$$

Together Eq.(35) and Eq.(37) give four equations which can be broken up into two systems of two equations. The unknowns in the equations are the amplitudes of the reflected and transmitted waves. Solving these systems is a matter of basic algebra and their solutions mirror one another. So we will write the solutions for a single system and ignore the other under the assumption that it can be easily found through a similar procedure. Presuming we know the initial wave amplitude $A_0$, the amplitude of the the reflected wave and the transmitted wave are (from solving the system given by the left equations in Eq.(35) and Eq.(37))

$$A_T = \frac{2k_1}{k_1 + k_2} A_0, \qquad B_R = \frac{k_1 - k_2}{k_1 + k_2} A_0. \tag{38}$$

It will prove more useful to write this result in terms of the mass density $\mu$ of each string. Dividing Eq.(38) by $\omega$ (which is the same on both sides of the string), and using $k/\omega = v^{-1}$, we find

$$A_T = \frac{v_1^{-1}}{v_1^{-1} + v_2^{-1}} A_0, \qquad B_R = \frac{v_1^{-1} - v_2^{-1}}{v_1^{-1} + v_2^{-1}} A_0, \tag{39}$$

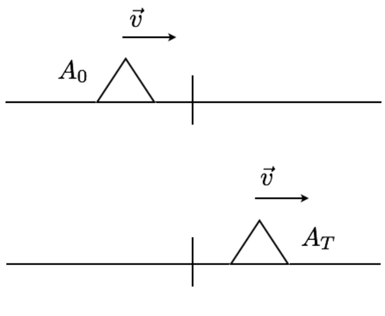


Figure 4: $\mu_2 = \mu_1$

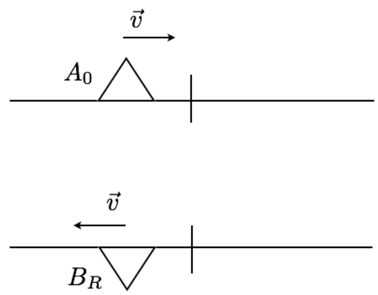


Figure 5: $\mu_2 \to \infty$

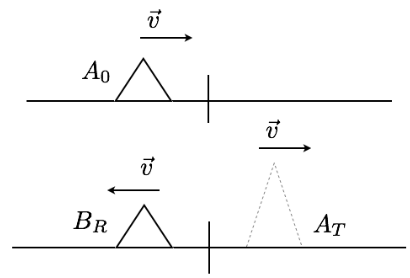


Figure 6: $\mu_2 \to 0$

and then using the definition of velocity as $v = \sqrt{T/\mu}$ (with $\mu$ different on both sides of the string, but $T$ the same by imposition) we find

$$A_T = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}A_0, \qquad B_R = \frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}A_0. \tag{40}$$

These are the final expressions relating the initial wave amplitude $A_0$ to the reflected amplitude $B_R$ and the transmitted amplitude $A_T$. We would find identical expressions relating $B_0$, $A_R$ and $B_T$ if we solved the system given by the right equations in Eq.(35) and Eq.(37). Most often, we define the coefficients of Eq.(40) as **reflection** and **transmission** coefficients

$$T = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}, \qquad R = \frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}} \tag{41}$$

Because these coefficients are real, they apply equally well when defining the relationships between the initial, reflected, or transmitted amplitude for the real traveling waves of the form Eq.(8). What properties do these equations lend to real traveling waves? We can best understand this through limiting cases concerning how the mass density of the string changes across the origin.

– $\boldsymbol{\mu_2 = \mu_1}$ **(Uniform String):** If we have a uniform string across the origin, then we expect no wave to be reflected back and that all of the initial wave is transmitted from the left to the right. With Eq.(40) we find exactly this, for with $\mu_1 = \mu_2$, Eq.(41) gives us $T = 1$ and $R = 0$, indicating no wave is reflected and the transmission amplitude matches the initial amplitude. (Fig. 4)

– $\boldsymbol{\mu_2 \to \infty}$ **(Infinitely heavy on right):** If we have an infinitely heavy string on the right, then this is tantamount to having the string fixed at $x = 0$. Eq.(41) then tells us that no wave is transmitted and the wave which is reflected has negative the amplitude of the initial wave. (Fig. 5)

  More generally, we see that whenever $\mu_2 > \mu_1$, the amplitude of the reflected wave $B_R$ always has a negative sign relative to the incident wave $A_0$. This means that if $A_0$ is positive, $B_R$ would be negative and vice versa; Summarily, whenever $\mu_2 > \mu_1$ the incident wave experiences a phase change of $\pi$ upon reflection (because $e^{i\pi} = -1$). Moreover, for $\mu_2 > \mu_1$, we $|R| > |T|$ meaning that when transitioning to a denser string, the amplitude of the reflected wave is greater than the amplitude of the transmitted wave. This makes sense seeing as it would be harder for a wave to propagate through a heavier string.

– $\boldsymbol{\mu_2 \to 0}$ **(Infinitely light on right):** If we have an infinitely light string on the right, then this is tantamount to having no string after $x = 0$. Eq.(41) tells us the transmitted amplitude is twice that of the initial amplitude, but this is fictitious because if $\mu_2 = 0$ then there is no string through which this amplitude can propagate. Thus all of the (real) wave returns to its source and we have $R = 1$. (Fig. 6)

More generally, for $\mu_2 < \mu_1$, we find $|R| < |T|$ indicating that when transitioning to a less dense string, the amplitude of the transmitted wave is greater than the amplitude of the reflected wave. This makes sense seeing as it would be easier for a wave to propagate through a lighter string.

## 3.4 Wave Attenuation

In our derivation of the wave equation, we analyzed the properties of coupled oscillators under the assumption that each oscillator experienced the energy-conserving spring force definitive of Hooke's law. This was also our basic assumption when we analyzed simple harmonic motion, but to bring our models closer to reality, we later incorporated the effects of air drag given the knowledge that mechanical energy is rarely conserved in actual systems. Similarly, to describe real waves we would need to incorporate such energy-dissipating effects into our derivation of the wave equation.

For the case of coupled masses oscillating in the transverse direction (i.e., a direction perpendicular to the axis which defined their couplings), our previous equation of motion for the mass in the $j$th position was

$$m\ddot{y}_j = k(y_{j+1} - y_j) - k(y_j - y_{j-1}). \qquad \text{[Energy-conserving system].} \tag{42}$$

If we were to incorporate air-drag into this equation, we would need to include the velocity dependent drag force $-b\dot{y}_j$. Doing so, gives us the equation of motion

$$m\ddot{y}_j = -b\dot{y}_j + k(y_{j+1} - y_j) - k(y_j - y_{j-1}). \tag{43}$$

Mirroring our previous derivation of the wave equation, we can take our lattice spacing $a$ to 0 and define macroscopic quantities like the linear mass density $\mu = \lim_{a\to 0} m/a$ and the average string tension $T = \lim_{a\to 0} ka$. We would also have to define a drag coefficient for the string itself as

$$\beta \equiv \lim_{a\to 0} \frac{b}{a}. \tag{44}$$

The resulting wave equation would be

$$\frac{\partial^2 y}{\partial t^2} + 2\lambda \frac{\partial y}{\partial t} = v^2 \frac{\partial^2 y}{\partial x^2}, \tag{45}$$

where $v = \sqrt{T/\mu}$ and $2\lambda = \beta/\mu$. To solve this wave equation, we can use the fact that the differential equation is "linear and homogeneous with constant coefficients" to apply methods similar to those used to solve the other equations of motion with similar properties. First we rewrite this equation in terms of the complex variable $z(x,t)$ (where $y(x,t) = \text{Re}[z(x,t)]$).

$$\frac{\partial^2 z}{\partial t^2} + 2\lambda \frac{\partial z}{\partial t} = v^2 \frac{\partial^2 z}{\partial x^2}, \tag{46}$$

Given its properties, we can assume that the solution to this equation is an exponential (representing a wave traveling to the right) of the form

$$z(x,t) = Ae^{i(kx-\omega t)}, \tag{47}$$

where the relationship between $k$ and $\omega$ has yet to be determined. Inserting this solution into Eq.(46), we find

$$-\omega^2 - i2\lambda\omega = -v^2 k^2, \tag{48}$$

which is the desired relationship between $k$ and $\omega$. Given Eq.(48), there can be various types of dissipative behaviors in our system and these behaviors are determined by our system's boundary conditions.

As a specific example, we follow the case outlined at the end of http://www.people.fas.harvard.edu/ djmorin/waves/transverse.pdf. Imagine that at $x = 0$, the string is oscillating with a wave amplitude $Ae^{-i\omega t}$ which persists for all time. Since there is no decay in the amplitude we know $\omega$ must be exclusively real and

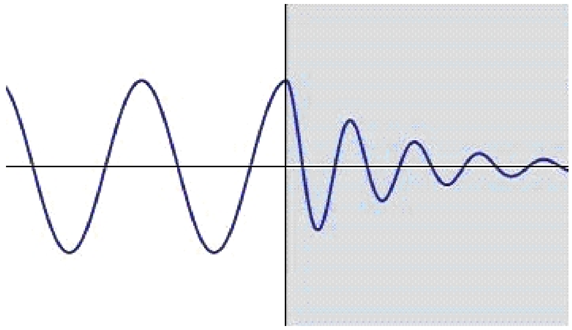

Figure 7: Wave amplitude decreasing after entering a dissipative medium. Figure from http://www.phy.pmf.unizg.hr/ mpozek/research.html

thus Eq.(48) must define a wave number $k$ which is complex. Solving for this wave number we find

$$k = \sqrt{\left(\frac{\omega}{v}\right)^2 - 2i\lambda} \equiv k_0 + iK, \tag{49}$$

where we used the fact that $\sqrt{a+ib}$ is a complex number to replace it with a real part $k_0$ and an imaginary part $K$[6]. We thus find that Eq.(47) becomes

$$z(x,t) = Ae^{-Kx}e^{i(k_0x-\omega t)}. \tag{50}$$

This is a single solution, the other solution could be found through a similar procedure by beginning with $y(x,t) = Be^{-i(kx-\omega t)}$. We would then find the general solution is

$$z(x,t) = Ae^{-Kx}e^{i(k_0x-\omega t)} + Be^{-Kx}e^{-i(k_0-\omega t)}. \tag{51}$$

Computing the real part of this quantity gives us the physical solution for this situation:

$$y(x,t) = \text{Re}[z(x,t)] = A_0e^{-Kx}\cos(k_0x - \omega t - \phi), \tag{52}$$

where $A_0$ and $\phi$ are phases set by the initial conditions of this system. Given what we know about how exponentials change the amplitude of waves, we should easily be able to visualize Eq.(52). It would consist of a wave decaying exponentially as a function of *distance* from the origin (See Fig. 7).

Compared to our previous traveling wave solutions, Eq.(52) better represents how real waves propagate because such waves are usually moving through media which dissipate the wave's initial energy.

# 4 Sound Waves

We have so far been discussing the properties of waves rather generally using the waves on a string as an example. What about the descriptions of the sound and light waves which served as the introduction to this chapter? In this final section, we will briefly discuss sound waves, leaving light (or electromagnetic waves) to the next lesson.

Sound waves (like waves on a string and unlike electromagnetic waves) comprise propagation of a disturbance through a medium. For sound waves the medium consists of a gas, liquid, or a solid, and the disturbances are longitudinal oscillations (i.e., oscillations along the direction of wave propagation) in the positions of the molecules which define the medium. We present the wave equation for sound waves, with-

[6]To solve for it explicitly we could use the identity $\sqrt{a+ib} = \rho(\cosh\eta + i\sinh\eta)$ where $\rho^2 = a$ and $2\eta = \sinh^{-1}(b/2a^2)$.

out derivation to highlight its similarity to the wave equations we've studied so far (derivations can be found in most of the references on the website). We will only consider (again for simplicity) sound waves propagating in a single direction.

Say we have a region of space consisting of a gas of molecules of mass density $\rho$. If the total pressure in that region of space is $P_{\text{tot}}$, we can separate $P_{\text{tot}}$ into a base pressure $P_0$ and a pressure fluctuation $p$, the latter of which leads to what we experience as a sound wave:

$$P_{\text{tot}} = P_0 + p. \tag{53}$$

Then considering the properties of the longitudinal oscillation and some results from thermal physics, we find that the wave equation for $p$ is

$$\frac{\partial^2 p}{\partial t^2} = \frac{\gamma P_0}{\rho} \frac{\partial^2 p}{\partial x^2}, \tag{54}$$

where $\gamma$ is a quantity which depends on the atomic composition of a molecule of the gas; for diatomic gases (like $N_2$ and $O_2$ which constitute much of air), $\gamma = 7/5$. Given our previous wave equation, we can infer that the speed of the wave propagation for pressure waves is

$$c = \sqrt{\frac{\gamma P_0}{\rho}}. \tag{55}$$

Given that, for air, $\gamma \approx 7/5$, $\rho = 1.275$ kg/cm$^3$, and $P_0 = 1$ atm, we find that the numerical value of the speed of the pressure wave is

$$c \approx 330 \text{ m/s}, \tag{56}$$

which is indeed the speed of sound.

# References

[1] H. Georgi and A. P. French, "The physics of waves," 1993.