Lecture 09: Maxwell's Equations and Electromagnetic Waves

In these notes, we review the basic phenomena of electromagnetism and show how Maxwell's equations lead to the wave equation for electromagnetic waves. We conclude by discussing some basic properties of electromagnetic waves.

1 Electricity, Magnetism, and Light

In the last lesson, we motivated our discussion of traveling waves by noting that we are able to hear sound because pressure waves propagate through the air from their sources to our ear drums. Similarly, we are able to see objects because electromagnetic waves propagate from their sources and into the rod and cone cells of our eyes. Both sound and light are wave phenomena and thus can be presumed to obey the wave equation we previously derived or at least one similar to it[1]. However, the connection between light and the propagating degrees of freedom responsible for its properties is not as clear as the connection between sound and pressure waves.

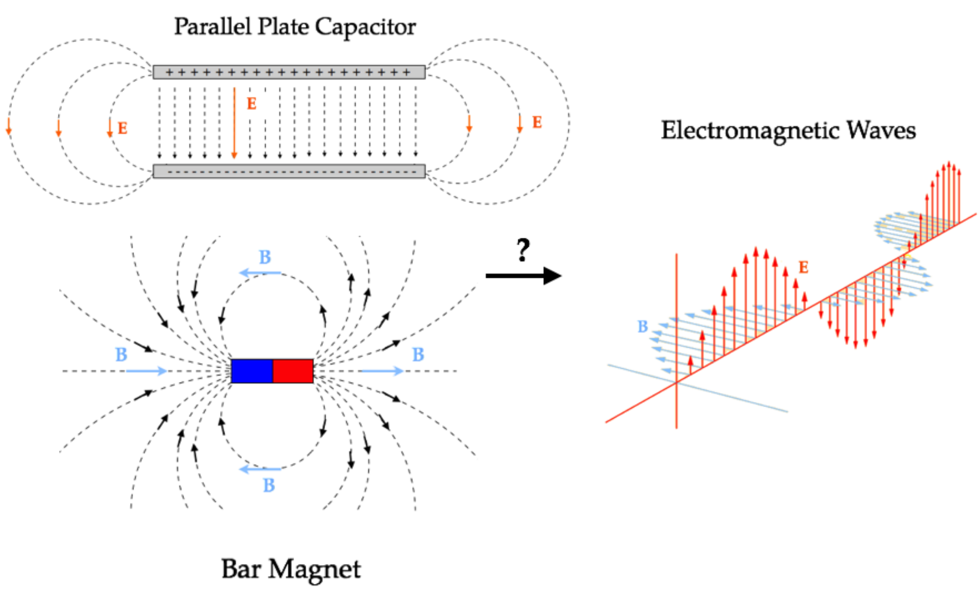

Figure 1: How are capacitors and bar magnets related to electromagnetic waves?

Our understanding of bar magnets and static electricity does not at all indicate that such phenomena are fundamentally related to the phenomena which allows us to see. If light truly consists of electromagnetic waves, how is that manifestation of electromagnetism related to the more pedestrian manifestations which power electronics and medical devices (like NMR machines). This was the problem James Clerk Maxwell faced (and solved) in the 19th century, and it is the problem we will discuss in these notes.

[1]This is not exactly true. Simply because a quantity propagates as a wave does not mean it obeys the specific wave equation we have been discussing. There are *many* different wave equations, and recognizing wave-like properties and then determining the specific equation governing those properties are two different things. Indeed, moving from the former to the latter is usually quite difficult.

Framing Question:
What is the physics of light waves? More, specifically, how are electromagnetic waves connected to electricity and magnetism?

2 Primer on Vector Calculus

Before we consider the physics of electromagnetism we need to complete a short review of the relevant mathematics.

Recalling the table we wrote down in Lecture notes 01, we know that the mathematics of electromagnetism consists of vector calculus and partial differential equations. We have already had some exposure to partial differential equations through our study of the wave equation and its solutions, and it turns out much of the methods we studied in that context can be applied to electromagnetism. Thus, what is left for us to study is the vector calculus which defines the equations of electromagnetism.

Table 1: Physical Theories: Principles and the Mathematics used to obtain Predictions

	Principles	Mathematics	Example Result/Prediction
Electromagnetism	Maxwell's Equations; Lorentz Force Law	Vector Calculus; Partial Differential Equations	Electric field of General Conductor; Speed of Light

The major results of vector calculus can be seen as an extension of the major result of single-variable calculus: The fundamental theorem of calculus. In one form, the theorem states

$$\int_a^b \frac{d}{dx}F(x)\,dx = F(b) - F(a) \tag{1}$$

In other words, integrating the derivative of a function from one point in the domain to another point is equal to the difference between the function's value at each of those points. Eq.(1) is all well and good for single-variable functions, but how would we extend it to consider functions of many variables or functions written as vectors? This is the question we will answer in the subsequent sections

Question:
How can we develop a "fundamental theorem of calculus" in higher dimensions? for multivariable functions? for functions which are vectors.

2.1 Scalar Functions and Vector Fields

To formulate generalizations of Eq.(1), we must first discuss generalizations of the concept of a function. By this point in your education, you should be very familiar with the function $y = f(x)$ which (if it is a valid function) returns a single **scalar** value for every single value x. We can easily generalize this function to a multivariable function $w = \phi(x, y, z)$ which returns a single scalar value for every value of (x, y, z). Namely, $\phi(x, y, z)$ is a function in which the independent variables consist of points in $\mathbb{R}^3$ (Euclidean space). Some examples of multivariable scalar functions are

$$\phi(x,y,z) = x + y + z, \qquad \phi(x,y,z) = x^2 y + \sin(4z), \qquad \phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}. \tag{2}$$

Besides defining scalars in Euclidean space, we know it is possible to define vectors as well. You are likely used to to considering constant vectors like $\vec{v} = (4,1,3)$ where 4 is the component of the vector along the x-axis, 1 the component along the y-axis, and 3 the component along the z-axis. We can write this vector in terms of unit vectors as

$$\mathbf{v} = (4,1,3) = 4\hat{\mathbf{x}} + \hat{\mathbf{y}} + 3\hat{\mathbf{z}}, \tag{3}$$

where $\hat{\mathbf{x}} = (1,0,0)$, $\hat{\mathbf{y}} = (0,1,0)$, and $\hat{\mathbf{z}} = (0,0,1)$ are the unit vectors in the x, y, and z direction, respectively. Such vectors are the same regardless of how we translate them or move them around in space. Now, when we have a vector whose value depends on the point in Euclidean space at which we evaluate the vector, then we have what is known as a **vector field.** A vector field $\mathbf{F}(x,y,z)$ is a vector which is a function of points (x,y,z) in space. In general, we take the vector to have the same number of components as it has arguments. Thus the vector field $\mathbf{F}(x,y,z)$ has three components F_x, F_y, F_z and each component is a function of x, y, and/or z. Writing this vector field in component form we have

$$\mathbf{F}(x,y,z) = (F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)) = F_x(x,y,z)\hat{\mathbf{x}} + F_y(x,y,z)\hat{\mathbf{y}} + F_z(x,y,z)\hat{\mathbf{z}}. \tag{4}$$

Some examples of vector fields are

$$\mathbf{F}(x,y,z) = (x,y,z), \qquad \mathbf{F}(x,y,z) = \sin(xz)\hat{\mathbf{x}} + \cos(y)\hat{\mathbf{z}}, \qquad \mathbf{F}(x,y,z) = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{3/2}}. \tag{5}$$

2.2 Derivative operators

With our definitions of scalar fields and vector functions, we can now move on to define the various derivative operators which can be applied to these fields. We will begin with the mathematical definition of these operators, and in the next section will use their integral theorems to understand them intuitively.

All of these derivative operators make use of a particular derivative operator cryptically[2] termed **nabla** or **del**. This operator is defined as

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}. \tag{6}$$

We note that nabla is composed of partial derivatives—the same kind of derivatives which appear in the wave equation. Just as a review, the partial derivative of a function with respect to a variable x is simply the derivative with respect to x with all other variables held constant. For example, the various partial derivatives of the function $\phi(x,y,z) = x^2y + \sin(4z)$ are

$$\frac{\partial \phi}{\partial x} = 2xy, \qquad \frac{\partial \phi}{\partial y} = x^2, \qquad \frac{\partial \phi}{\partial z} = -4\cos(4z). \tag{7}$$

When we apply Eq.(6) directly to a scalar function, it is said we are taking the **gradient** of the scalar function. From Eq.(7), we see that applying ∇ to $\phi(x,y,z)$ yields

$$\nabla\phi = (2xy, x^2, -4\cos(4z)) = 2xy\,\hat{\mathbf{x}} + x^2\,\hat{\mathbf{y}} - 4\cos(4z)\,\hat{\mathbf{z}} \tag{8}$$

We note then that applying the ∇ operator to a scalar function ϕ results in a vector field $\nabla\phi$.

Our second derivative operator is called **the divergence** and it is only applied to vector fields. The di-

[2]Wikipedia tells me the term ”nabla” comes from the Hellenistic Greek word for ”harp” which is apparently what ∇ looks like.

vergence of the arbitrary vector field Eq.(4) is denoted by $\nabla \cdot \mathbf{F}$ and is defined as

$$\nabla \cdot \mathbf{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{9}$$

For example, the divergence of $\mathbf{F} = x\,\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ is

$$\nabla \cdot \left(x\,\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\right) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3. \tag{10}$$

We note that when we took the divergence of the vector field in this case, we obtained a scalar number; in general, taking the divergence of a vector field yields a scalar function.

Our third derivative operator is called **the curl** and it also only can be applied to vector fields. The curl of the arbitrary vector field Eq.(4) is denoted by $\nabla \times \mathbf{F}$ and is defined as

$$\nabla \times \mathbf{F} = \hat{\mathbf{x}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right). \tag{11}$$

More compactly we can think of $\nabla \times \mathbf{F}$ as the cross-product between ∇ and $\mathbf{F}$. In this case, we still get Eq.(**??**) as a definition, but we can write the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}. \tag{12}$$

For example, computing the curl of the $\mathbf{F}(x,y,z) = \sin(xz)\hat{\mathbf{x}} + \cos(y)\hat{\mathbf{z}}$ gives us

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin(xz) & 0 & \cos(y) \end{vmatrix} = -\sin(y)\,\hat{\mathbf{x}} - x\cos(xz)\,\hat{\mathbf{y}}. \tag{13}$$

We note that, by Eq.(12), taking the curl of a vector field yields another vector field.

The final differential operator which will be relevant for our study of Maxwell's equations is the **Laplacian.** The Laplacian, denoted by ∇^2 is defined as the divergence of the gradient and thus is a scalar differential operator. For example, the Laplacian of a field ϕ is

$$\nabla^2\phi = \nabla \cdot \nabla\phi = \nabla \cdot \left(\frac{\partial \phi}{\partial x}\,\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\,\hat{\mathbf{y}} + \frac{\partial \phi}{\partial z}\,\hat{\mathbf{z}}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}. \tag{14}$$

Thus we can generally define the Laplacian as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{15}$$

The Laplacian (because it is a scalar operator) can also be applied to vector fields.

We can apply these derivative operators in various combinations to scalar functions and vector fields, and the various ways we can apply these operators are associated with various identities in vector calculus. For example, it's possible to show that for an arbitrary function ϕ that the curl of the gradient is zero:

$$\nabla \times (\nabla\phi) = 0. \tag{16}$$

And for an arbitrary vector field **F** the divergence of the curl is zero:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0. \tag{17}$$

A final identity which will be needed to derive the electromagnetic wave equation involves the curl of a curl. That is, for an arbitrary vector field **F**, we can show

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}. \tag{18}$$

2.3 Theorems of Vector Calculus

With our above outlined derivative operators, we are now prepared to answer the question posed at the beginning of this section: How can we develop a "fundamental theorem of calculus" in higher dimensions/for scalar functions/for vector fields? If you take a course in multivariable calculus, you will spend the entire semester learning about such generalizations, but for our purposes we will only state the results.

The first result is called **the fundamental theorem of line integrals** or **the gradient theorem**:

$$\int_{\mathbf{r}_0}^{\mathbf{r}_f} d\ell\, \hat{\boldsymbol{\ell}} \cdot \nabla \phi = \phi(\mathbf{r}_f) - \phi(\mathbf{r}_0), \tag{19}$$

where $\mathbf{r}_f = (x_f, y_f, z_f)$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$ denote the final and initial positions, respectively, of a curve in Euclidean space. The vector $\hat{\boldsymbol{\ell}}$ is a unit vector tangential to the curve along a point in the integration.

Next, we have **the divergence theorem**:

$$\int_V dV\, \nabla \cdot \mathbf{F} = \int_S dA\, \hat{\mathbf{n}} \cdot \mathbf{F}, \tag{20}$$

where V defines a volume in Euclidean space and S is the boundary of that volume (for example, if V is the volume of a sphere, then S is the surface of the sphere). The vector $\hat{\mathbf{n}}$ is a unit vector normal to the surface S at a point in the integration.

Finally, we have **Stokes' theorem (for 3D space)**

$$\int_S dA\, \hat{\mathbf{n}} \cdot \nabla \times \mathbf{F} = \oint_\Gamma d\ell\, \hat{\boldsymbol{\ell}} \cdot \mathbf{F}, \tag{21}$$

where S defines a surface in Euclidean space and Γ denotes the boundary of that surface (for example, if S is the area of a circle, then Γ is the perimeter of the circle). The vectors $\hat{\mathbf{n}}$ and $\hat{\boldsymbol{\ell}}$ are defined similarly to their definitions in Eq.(20) and Eq.(19). The symbol $\oint$ denotes the fact that we are integrating over a "closed curve" (i.e., one where the endpoints are the same as the starting points) rather than an open curve.

As for terminology, we term the surface integral on the right-hand side of Eq.(20) the **flux** of the vector field through the defined surface, and we term the line integral on the right-hand side of Eq.(21) the **circulation** of the vector field over the defined curve:

$$\text{Flux through } S = \int_S dA\, \hat{\mathbf{n}} \cdot \mathbf{F}, \tag{22}$$

$$\text{Circulation around } \Gamma = \oint_\Gamma d\ell\, \hat{\boldsymbol{\ell}} \cdot \mathbf{F} \tag{23}$$

2.4 Meaning of Derivative Operators

From the above theorems, we can develop some intuition for the meanings of the previously defined derivative operators. First, we begin with the fundamental theorem of calculus:

$$\int_{x_0}^{x_f} dx\, \frac{dF}{dx} = F(x_f) - F(x_o) \tag{24}$$

Presuming we didn't already have such a definition, we could use Eq.(24) to establish an intuitive definition of the derivative. If we take the domain of points over which we're integrating to be very small (i.e., take $|x_f - x_0|$ to be very small), then we can approximate the integral as dF/dx evaluated at the point x_0 (or x_f) multiplied by the space of integration $x_f - x_0$.

$$F(x_f) - F(x_o) = \int_{x_0}^{x_f} dx\, \frac{dF}{dx} \simeq (x_f - x_0)\frac{dF(x_0)}{dx} \tag{25}$$

This approximation gets better and better the closer x_f is to x_0. In fact, dividing Eq.(25) by $x_f - x_0$ and taking the limit as $x_f \to x_0$ should give us an exact result. Therefore, we find

$$\lim_{x_f \to x_0} \frac{F(x_f) - F(x_0)}{x_f - x_0} = \frac{dF(x_0)}{dx}. \tag{26}$$

Eq.(26) establishes what we already knew about derivatives: derivatives are the change in a function divided by the change in the argument of the function at a specific point, or, less verbosely, they are the instantaneous rate of change in the function.

We can use similar procedure to establish intuitive definitions of the gradient, divergence, and curl operators. The definition of the gradient is most similar to the definition of the derivative. From Eq.(19), we have

$$\lim_{\ell \to 0} \frac{\phi(\mathbf{r}_0 + \ell\,\hat{\boldsymbol{\ell}}) - \phi(\mathbf{r}_0)}{\ell} = \hat{\boldsymbol{\ell}} \cdot \nabla\phi(\mathbf{r}_0), \tag{27}$$

where $\boldsymbol{\ell} = \ell\,\hat{\boldsymbol{\ell}}$, with ℓ being the length of the vector $\boldsymbol{\ell}$ and $\hat{\boldsymbol{\ell}}$ being the unit vector in the direction of $\boldsymbol{\ell}$. We thus find that, like the derivative, the gradient of a function at a point is the rate of change of that function at that point, except with one difference from the derivative: the gradient has direction. Computing the dot product between the gradient and a unit vector $\hat{\boldsymbol{\ell}}$ gives us the rate of change in the *specific direction* defined by $\hat{\boldsymbol{\ell}}$.

Using Eq.(20), we can find that the divergence at a point $\mathbf{r}_0$ is equal to

$$\nabla \cdot \mathbf{F}(\mathbf{r}_0) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_S d\mathbf{A} \cdot \mathbf{F} = \text{flux volume-density}, \tag{28}$$

where ΔV is a small volume-element surrounding $\mathbf{r}_0$ and S is the surface area of that volume element. From the definition of flux in Eq.(22), we see that the divergence of a vector field at a point is the flux density (for volume) of the field at that point.

Similarly, using Eq.(21), we find that the curl at a point $\mathbf{r}_0$ and in a direction $\hat{\mathbf{n}}$ is

$$\nabla \times \mathbf{F}(\mathbf{r}_0) \cdot \hat{\mathbf{n}} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_\Gamma d\ell\,\hat{\boldsymbol{\ell}} \cdot \mathbf{F} = \text{circulation area-density}, \tag{29}$$

where ΔA is a small area-element surrounding $\mathbf{r}_0$, $\mathbf{n}$ is a unit vector which is normal to that area element, and Γ is the closed curve forming the boundary of that area. From the definition of circulation in Eq.(23), we see that the curl of a vector field at a point is the circulation density (for area) of the field at that point. Like the gradient, the curl also has direction, and thus the definition Eq.(29) gives the circulation density in the direction $\hat{\mathbf{n}}$, the normal vector to the surface ΔA.

This was a very quick and cursory run through of the major results of vector calculus, so don't worry if you don't understand everything just mentioned. It takes time and many examples to develop a conceptual understanding of what gradients, divergences, and curls actually represent, but we will mostly be using these derivative operators in a mathematical capacity.

3 Maxwell's Equations

As shown in Table 1, the principles of electromagnetic theory are codified into a set of equations called Maxwell's equations and an additional equation termed the Lorentz force law. The Lorentz force law tells us how electric and magnetic fields affect the dynamics of charged particles. Here, we will not be concerned with the dynamics of charged particles, so we will be focusing on Maxwell's equations. We will present and discuss each one in turn.

- **Gauss's Law of Electric fields**

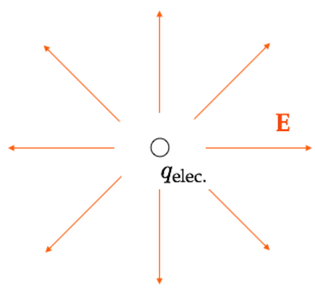

Figure 2

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{30}$$

In Eq.(30), $\mathbf{E}$ is the electric field, ρ is the electric charge density, and ϵ_0 is a physical constant. This law determines how charge distributions ρ create electric fields which extend outward from the location of the distribution. The manifestation of this law you're likely most familiar with is magnitude of the electric field $\mathbf{E}$ at a point $\mathbf{r}$ created by a charged particle Q at a point $\mathbf{r}_0$:

$$|\mathbf{E}(\mathbf{r})| = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_0|^2}. \tag{31}$$

Using the integral form of Eq.(30), it is possible to derive Eq.(33) in addition to the electric fields for many general charge distributions.

- **Gauss's Law of Magnetic fields**

$$\nabla \cdot \mathbf{B} = 0 \tag{32}$$

Eq.(32) basically states that there is no charged particle which creates magnetic fields in the same way that electric charges create electric fields. Such a charged particle would be termed a magnetic monopole, and as far as current experimental physics is concerned such a particle is entirely conjectural.

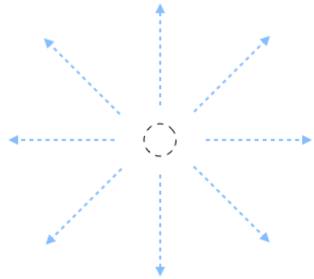

Figure 3

$$|\mathbf{B}(\mathbf{r})| \neq \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_0|^2} \qquad \text{[Magnetic monopoles do not exis].} \tag{33}$$

- **Faraday's Law**

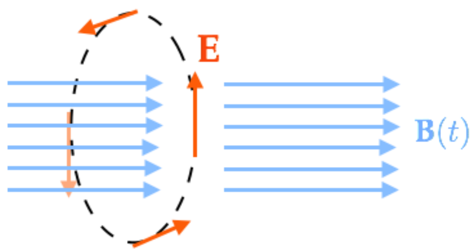

Figure 4: Bar magnet moving to the right generates an opposing electric field in a loop.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{34}$$

Eq.(34) states that a changing magnetic field induces an electric field. The negative sign is significant here because it indicates the electric field is generated such that it creates a current which opposes the magnetic field. You might have seen Eq.(34) written in integral form in which it states that a changing magnetic flux (by the definition of flux in Eq.(22)) generates an electric potential difference around a closed loop. This law expresses the basic physics of all electric generators (i.e., devices which convert mechanical energy into electrical energy) and thus underlies much of the electronic infrastructure of the late 19th century until today.

- **Ampère- Maxwell Law**

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{35}$$

In Eq.(35), **J** is the electric current area-density, and μ_0 is the magnetic permeability, a physical constant. Eq.(35) can be seen as a counterpart to both Eq.(30) and Eq.(34). Namely, it states how magnetic fields are generated both by electric charge current densities and by changing electric fields. The phenomena underlying the first term in Eq.(35) were first understood by Ampère and thus this part of the equation is often called "Ampère's Law." You might have seen this law expressed for current wires as

$$|\mathbf{B}(\mathbf{r})| = \frac{\mu_0 I}{2\pi|\mathbf{r}-\mathbf{r}_0|} \tag{36}$$

Figure 5 Figure 6

Figure 7: Magnetic field created by a current carrying wire in Fig. 5 and by a changing electric field Fig. 6.

where I is the current and $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the current carrying wire and the point where we evaluate the magnetic field. The second term was postulated by Maxwell in his treatise on electricity and magnetism [1]. Before Maxwell, physicists and mathematicians working in electricity and magnetism knew that changing magnetic fields could generate electric fields, but they did not believe the reverse could happen. Maxwell's addition turns out to be the crucial factor in explaining why the basic phenomena of electricity and magnetic are related to light (i.e., electromagnetic) waves.

4 Deriving Electromagnetic Wave Equation

Now that we have collected the basic equations of electromagnetism, demonstrating that these equations imply the existence of electromagnetic waves is quite easy. First, we specify we want to study electric and magnetic fields in free space far from any distributions of particles which could change their self-determining dynamics. Such conditions would well model the contexts (such as outer space or across long distances) in which we expect electromagnetic waves to propagate far from their sources.

Now, if we were to study these equations in locations of space very far from the charge distribution ρ and the current density $\mathbf{J}$, then we could set these charge and current terms to be zero in their respective equations. The resulting set of equations are called the **source-free Maxwell Equations**:

$$\nabla \cdot \mathbf{E} = 0 \tag{37}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{38}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{39}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{40}$$

It is with these equations that we can derive the desired equation describing electromagnetic waves. Taking the curl of Eq.(34), and using the identity Eq.(18) we find

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= -\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) \\
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\frac{\partial}{\partial t} \nabla \times \mathbf{B} \\
-\nabla^2 \mathbf{E} &= -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},
\end{aligned} \tag{41}$$

where we used the fact that spatial and partial derivatives commute in the second line and Eq.(40) and Eq.(37) in the last line. By a similar calculation, taking the curl of Eq.(40) gives us an identical result for the magnetic field. All together, the source free Maxwell equations imply that electric and magnetic fields far from their sources obey the equations

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{42}$$

With Eq.(42) we have achieved our task of connecting basic electric and magnetic phenomena to electromagnetic waves. Eq.(42) are **three-dimensional** wave equations for the electric and magnetic fields. Thus, the two equations have two features that our previous wave equations did not: one, the propagating degree of freedom is a vector field, not a scalar function; two, the wave is propagating in three dimensions instead of two. How these two new features affect the solutions to Eq.(42) can actually be easily extrapolated from our understanding of one-dimensional waves of scalar functions as we will discuss below.

But first we note one result which can easily be inferred from Eq.(42), and was the first model-based evidence that light consisted of propagating electric and magnetic waves. By dimensional analysis and our understanding of wave equations, we can expect that the coefficient on the right-hand side of the equations in Eq.(42) is the inverse-speed squared of the wave. Both the physical constants μ_0 and ϵ_0 were experimentally determined in contexts having nothing to do (ostensibly) with wave phenomena and thus their appearance in the above wave equations predicts the speed at which these electromagnetic waves should propagate. In SI units we have $\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$ and $\mu_0 = 4\pi \times 10^{-7}\ \text{N}/\text{A}^2$. Therefore Eq.(42) implies that the speed of wave propagation for electromagnetic waves is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \approx 3.00 \times 10^8\ \text{m/s}. \tag{43}$$

This was exactly the value of the speed of light which was experimentally determined decades before Maxwell consolidated the equations of electromagnetism[3]. Thus Maxwell's work provided a theoretical model for why light has the speed it does.

5 Basic properties of Electromagnetic Waves

In this section we explore the properties of the solutions to Eq.(42). We begin by guessing (and affirming) a solution to these equations and then we consider what Maxwell's equations imply about the nature of this solution. We begin with the guess (which is always reasonable for solutions to the basic wave equation) of a sinusoidal function for the **E** wave equation:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{44}$$

First checking that this equation satisfies the wave equation, we have

$$\begin{aligned} \nabla^2 \mathbf{E} &= \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ -\mathbf{k}^2\, \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) &= -\omega^2 \mu_0 \epsilon_0 \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \\ \mathbf{k}^2 &= \omega^2 \mu_0 \epsilon_0 \end{aligned} \tag{45}$$

Thus Eq.(44) is a solution as long as

$$\frac{1}{\mu_0 \epsilon_0} \equiv c^2 = \frac{|\mathbf{k}|^2}{\omega^2}. \tag{46}$$

In Eq.(44), $\mathbf{E}_0$ is a constant vector defining the amplitude of the wave. The vector **k** is called the wavevector and it is a vector analog to the wave number of one-dimensional systems. The wave vector defines the direction of propagation of the wave. For example, if **k** only had a z component, the wave would only vary

[3]See Fizeau-Foucault apparatus for a discussion of how this speed was experimentally determined in the mid 1850s.

in the z direction and hence we would say it propagates along the z axis. The vector $\mathbf{r}$ is defined as $\mathbf{r} = (x, y, z)$ and it simply denotes the position at which we evaluate the wave. Finally ω is the angular frequency of the wave motion.

To determine the important properties of Eq.(44), we need to return to Maxwell's equations. Substituting Eq.(44) into Eq.(37), we find

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{E} \\ &= \nabla \cdot (\mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)) \\ &= \mathbf{E}_0 \cdot \mathbf{k} \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \end{aligned} \tag{47}$$

from which we can infer $\mathbf{E}_0 \cdot \mathbf{k} = 0$. Given that $\mathbf{k}$ defines the direction of wave propagation, this result implies that the amplitude of the electric field is always perpendicular to the direction the wave is propagating. Beginning from a magnetic field expression similar to Eq.(44), we would similarly find $\mathbf{B}_0 \cdot \mathbf{k} = 0$. **Thus, one fundamental property of electromagnetic waves is that they are transverse waves and thus always have amplitudes orthogonal to their direction of propagation.**

Also substituting Eq.(44) into Eq.(39), we find

$$\begin{aligned} -\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E} \\ &= \nabla \times (\mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)) \\ &= (\mathbf{k} \times \mathbf{E}_0) \cos(\mathbf{k} \cdot \mathbf{r} - \omega t). \end{aligned} \tag{48}$$

Integrating, both sides of this result with respect to time (and dropping the constant of integration because it is not physically important), we have

$$\mathbf{B} = \mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t), \tag{49}$$

where we defined

$$\mathbf{B}_0 \equiv \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}_0) = \frac{1}{c}(\hat{\mathbf{k}} \times \mathbf{E}_0), \tag{50}$$

with $\hat{\mathbf{k}}$ being the unit vector in the direction of k. **Therefore, given the properties of cross products, we see that another fundamental property of electromagnetic waves is that propagating electric fields are always associated with propagating magnetic fields, and these two fields are perpendicular to both each other and the direction of propagation.** We depict both of these properties Fig. 8.

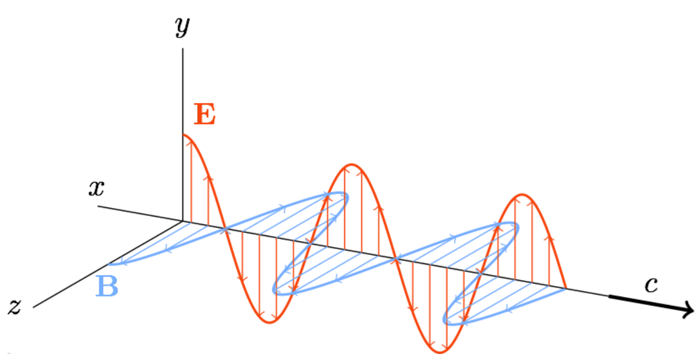

Figure 8: Electromagnetic waves propagating along the z axis. The two important properties of electromagnetic waves is that the electric and magnetic field parts of the wave are perpendicular to each other and to the direction of propagation.

References

[1] J. C. Maxwell, *A treatise on electricity and magnetism*, vol. 1. Clarendon press, 1881.