Physics III: Midterm –Solutions

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(Friday June 30)

First and Last Name: ________________________________

Exam Instructions:
This is an open-notebook exam, so feel free to use the notes you have transcribed throughout the summer and problem sets you have completed, but cellphones, laptops, and any notes written by someone else are prohibited.

Since this is a timed exam, your solutions need not be as ”organized” as are your solutions to assignments. Short calculations and succinct explanations are acceptable, and you can state (without derivation) the standard solutions to the equations of motion we derived in class. However, you should also recognize that you cannot receive partial credit for derivations/explanations you do not provide.

Also, **there are 75 possible points on this exam, but the exam will only be graded out of 60 points.** Thus, be mindful of the time you're spending on each problem.

Problem 1: ________________

Problem 2: ________________

Problem 3: ________________

Problem 4: ________________

Total:

1. **Short Calculation Questions (15 points)**
Answer the following questions, including a calculation justifying your answer. (You don't need to provide an elaborate explanation)

(a) (2 points) A vertical spring on which is hung a block of mass m_1 oscillates with angular frequency ω. With an additional block of mass $m_2 \neq m_1$ added to the spring, the frequency is $\omega/2$. What is the ratio m_1/m_2?

(b) (2 points) Two horizontal springs have identical spring constants, but one has a ball of mass m attached to it and the other has a ball of mass $2m$ attached to it. If the energies of the two systems are the same, what is the ratio of the oscillation amplitudes?

(c) (2 points) You stop your car to pick up a member of your car pool. After she gets in, does the angular frequency ω of the oscillation due to the car's suspension, increase, decrease or stay the same?

(d) (2 points) Given an object suspended by a spring, which of these variables of the motion can you control by varying the initial conditions: period, amplitude, energy of the system, frequency, phase, maximum velocity, maximum acceleration

(e) (2 points) We have the potential energy function

$$U(x) = B\left(\frac{x}{A} + \frac{A}{x}\right), \tag{1}$$

where x defines distance. What are the units of B? What are the units of A?

(f) (2 points) For the above $U(x)$: At what value of x would a particle of mass m in the potential be at a stable equilibrium?

(g) (3 points) For the above $U(x)$: The particle begins at rest a distance d away from the equilibrium position, where d is sufficiently small that simple harmonic motion is possible. The particle's initial energy is E_0. What would the energy be if A is changed to $2A$?

Solutions:

(a) We know the general formula for the angular frequency of a mass-spring system of spring constant k and total mass M_{tot} is

$$\omega = \sqrt{\frac{k}{M_{\text{tot}}}}. \tag{2}$$

If we have $\omega_{\text{before}} = \sqrt{k/m_1}$ and $\omega_{\text{after}} = \sqrt{k/(m_1+m_2)}$ and

$$\omega_{\text{after}} = \frac{1}{2}\omega_{\text{before}}, \tag{3}$$

then we have

$$\sqrt{\frac{k}{m_1+m_2}} = \frac{1}{2}\sqrt{\frac{k}{m_1}}, \tag{4}$$

or $m_1 + m_2 = 4m_1$ which gives us $\boxed{m_1/m_2 = 1/3}$.

(b) For a mass-spring system of spring constant k and amplitude of oscillation A, the total energy is $E = \frac{1}{2}kA^2$ which is independent of the mass. **If we have two identical springs (i.e., identical spring constants) oscillating with the same amplitude then their energies are the same.**

(c) We can effectively take the suspension of the car to be a spring of some spring constant k. If M_{tot} is the mass of the car then the angular frequency of the suspension is

$$\omega_0 = \sqrt{\frac{k}{M_{\text{tot}}}}. \tag{5}$$

If we add m to the M_{total} we find a new angular frequency

$$\omega_f = \sqrt{\frac{k}{M_{\text{tot}} + m}} < \sqrt{\frac{k}{M_{\text{tot}}}} = \omega_0. \tag{6}$$

Therefore, the angular frequency due to the car's suspension decreases.

(d) For a mass-spring system, the angular frequency is $\omega = \sqrt{k/m}$ and the period is $T = 2\pi/\omega$. Both of these quantities are fixed by the setup of the system and are therefore independent of the initial conditions. Everything else in the system is determined by the initial conditions. In particular, the motion is given by

$$x(t) = A_0 \cos(\omega t - \phi), \tag{7}$$

where $A_0 = \sqrt{x_0^2 + v_0^2/\omega^2}$ and $\tan\phi = v_0/\omega x_0$. The total energy of the system is $E = \frac{1}{2}kA_0^2$. The maximum velocity and acceleration are, respectively, $A_0\omega$ and $A_0\omega^2$. Thus we see that the **energy of the system, the phase, the maximum velocity, and the maximum acceleration** are controlled by the initial conditions.

(e) Because $U(x)$ has units of energy (i.e., Joules) and x has units of distance (i.e., meters), B **must have units of energy and** A **must have units of meters.**

(f) Computing $U'(x)$, and setting the result to zero we have

$$\begin{aligned} 0 &= U'(x) \\ &= B\left(\frac{1}{A} - \frac{A}{x^2}\right) \\ \frac{1}{A} &= \frac{A}{x^2} \end{aligned} \tag{8}$$

Therefore the equilibrium occurs at $x = \pm A$. We note that only the $\boxed{x = +A}$ equilibrium is stable because

$$U''(x) = \frac{2BA}{x^3} \quad \longrightarrow \quad U''(x = +A) = \frac{2B}{A^2} > 0. \tag{9}$$

Since $U''(A) > 0$, the potential energy has a local minimum at $x = A$.

(g) The energy of an oscillator of mass m, angular frequency ω and amplitude d is $E = \frac{1}{2}m\omega^2 d^2$. For the potential above the angular frequency is

$$\omega = \sqrt{\frac{U''(x = A)}{m}} = \sqrt{\frac{2B}{mA^2}}. \tag{10}$$

Thus the total energy is

$$E_0 = \frac{1}{2}m\frac{2B}{mA^2}d^2 = \frac{B}{A^2}d^2. \tag{11}$$

If we were to take A to $2A$, then E_0 would become $E_0/4$.

2. **An Angled Rail (20 points)**
Say we have a particle m constrained to move along a rail that makes a fixed angle θ with the vertical. The particle is attached to a spring of spring constant k which is itself attached to a wall by a sliding attachment such that the attachment is always at the same height as the mass and the spring is always horizontal. Say the particle is at a stable equilibrium when it is at the position shown in Fig. 1.

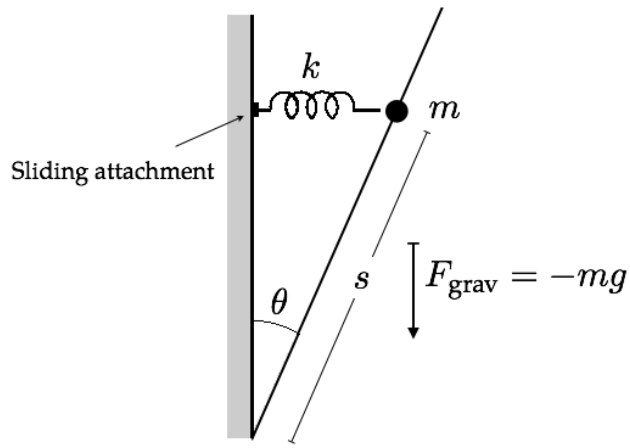

Figure 1: Angled Rails

We track the position of the particle with $s(t)$, the distance between the particle and the wall-rail attachment point.

(a) (5 points) If we set $\theta = \pi/2$, the mass only moves horizontally. In such a scenario, the mass has an equilibrium position $s = X_{\text{eq}}$. What is the equilibrium position for the general θ shown in the figure?

(b) (5 points) What is the equation of motion of the system in terms of s? What should the equation of motion be if we take $\theta \to 0$? Check that your two answers are consistent.

(c) (10 points) Say the particle is displaced from the equilibrium shown in Fig. 1 such that its total energy is E_0 and it first reaches the amplitude of its motion at time $t = 3/4\omega_0$ (where ω_0 is the angular frequency of oscillation). Find $s(t)$ as a function of time in terms of these initial conditions and the prior defined parameters.

Solutions:

(a) For $\theta = \pi/2$, the mass only moves horizontally and thus the vertical gravitational force is irrelevant. The net force exerted on the mass in this scenario is

$$F_{\text{net}} = -k(s - X_{\text{eq}}), \tag{12}$$

given that X_{eq} is the equilibrium position. By resolving the components of the spring and gravitational forces which lie along the rails, for general θ, the net force is

$$F_{\text{net}} = -k(s - X_{\text{eq}}) \sin\theta - mg\cos\theta. \tag{13}$$

Setting the right hand side to 0 for $s = s_{\text{eq}}$ and solving, we find

$$\boxed{s_{\text{eq}} = X_{\text{eq}} - \frac{mg}{k}\cot\theta.} \tag{14}$$

(b) By Newton's 2nd law and Eq.(13), the equation of motion of this system is

$$\boxed{m\ddot{s} = -k(s - X_{\text{eq}})\sin\theta - mg\cos\theta.} \tag{15}$$

If we take $\theta \to 0$, there should be no component of the spring force which lies along the rail. Therefore, the mass is only affected by a vertical gravitational force and we should have free falling motion $\boxed{\ddot{s} = -g.}$ Setting $\theta = 0$ in Eq.(15), we indeed find this to be the case.

(c) The equation of motion of the system Eq.(15) can be written as

$$m\ddot{u} = -ku, \tag{16}$$

where $u = s - s_{\text{eq}}$. The solution to this equation of motion is

$$u(t) = A_0\cos(\omega t - \phi). \tag{17}$$

By $E_0 = \frac{1}{2}kA_0^2$, and the fact that Eq.(17) reaches its first maximum when $t = 3/4\omega_0$, we find

$$A_0 = \sqrt{\frac{2E_0}{k}}, \quad \text{and} \quad \phi = 3/4. \tag{18}$$

Therefore, with $u(t) = s(t) - s_{\text{eq}}$ and s_{eq} defined in Eq.(14), we find

$$\boxed{s(t) = \sqrt{\frac{2E_0}{k}}\cos(\omega t - 3/4) + X_{\text{eq}} - \frac{mg}{k}\cot\theta.} \tag{19}$$

3. **Maximum speed (20 points)**
A mass on the end of a spring (with natural frequency ω_0) is released from rest at position x_0. The experiment is repeated, but now with the system immersed in a fluid that causes the motion to be damped (with damping coefficient $\gamma = b/2m$).

In terms of the above defined parameters:

(a) (5 points) For the case of critically damped motion, what is $x(t)$?

(b) (5 points) Again for the case of critically damped motion, at what time is the speed maximum ?

(c) (10 points) For the case of *overdamped* motion, what is $x(t)$?

Solution:

(a) For critically damped motion, the general solution to the damped oscillator equation of motion is

$$x(t) = (A + Bt)e^{-\gamma t}. \tag{20}$$

If the mass is released from rest at x_0, then we have the constraints

$$x_0 = A, \tag{21}$$
$$0 = -\gamma A + B. \tag{22}$$

Thus, we find $A = x_0$ and $B = \gamma x_0$. The specific solution is therefore

$$\boxed{x(t) = x_0(1 + \gamma t)e^{-\gamma t}.} \tag{23}$$

(b) Computing the velocity of this system, we find

$$\dot{x}(t) = x_0\gamma e^{-\gamma t} - x_0(1 + \gamma t)\gamma e^{-\gamma t} = -x_0\gamma t e^{-\gamma t}. \tag{24}$$

Computing the acceleration, gives us

$$\ddot{x}(t) = x_0\gamma^2 t e^{-\gamma t} - x_0\gamma e^{-\gamma t} = x_0\gamma(\gamma t - 1)e^{-\gamma t}. \tag{25}$$

Setting this acceleration to zero (as a condition for finding the maximum velocity), we find

$$\boxed{t = \frac{1}{\gamma}.} \tag{26}$$

(c) For overdamped motion, the general solution to the equation of motion is

$$x(t) = \left(Ae^{-(\gamma+\Gamma)t} + Be^{-(\gamma-\Gamma)t}\right), \tag{27}$$

where $\Gamma = \sqrt{\gamma^2 - \omega_0^2}$. If the mass is released from rest at x_0, then we have the constraints

$$x_0 = A + B \tag{28}$$
$$0 = A(-\gamma + \Gamma) + B(-\gamma - \Gamma). \tag{29}$$

Inserting the first equation into the second equation gives us the new system

$$x_0 = A + B \tag{30}$$
$$\frac{\gamma}{\Gamma}x_0 = A - B, \tag{31}$$

which yields the following solutions for A and B:

$$A = \frac{x_0}{2}\left(1 + \frac{\gamma}{\Gamma}\right), \quad \text{and } B = \frac{x_0}{2}\left(1 - \frac{\gamma}{\Gamma}\right). \tag{32}$$

We therefore find that the specific solution for $x(t)$ is

$$\boxed{x(t) = \frac{x_0}{2}\left[\left(1 + \frac{\gamma}{\Gamma}\right) e^{-(\gamma+\Gamma)t} + \left(1 - \frac{\gamma}{\Gamma}\right) e^{-(\gamma-\Gamma)t}\right].} \tag{33}$$

or

$$\boxed{x(t) = x_0 e^{-\gamma t}\left[\cosh(\Gamma t) + \frac{\gamma}{\Gamma}\sinh(\Gamma t)\right].} \tag{34}$$

4. **Forced Pendulum (20 points)**
Say we have an undamped pendulum consisting of a mass m attached to a string ℓ. The pendulum begins at rest from its equilibrium position, and at this initial time we apply a sinusoidal force $F(t) = F_0\cos(\omega t)$ to the mass. This force is applied along the arc of the mass's trajectory that is, the force is tangential to the arc and has no radial components (as shown in Fig. 2).

Take the small-angle approximation throughout.

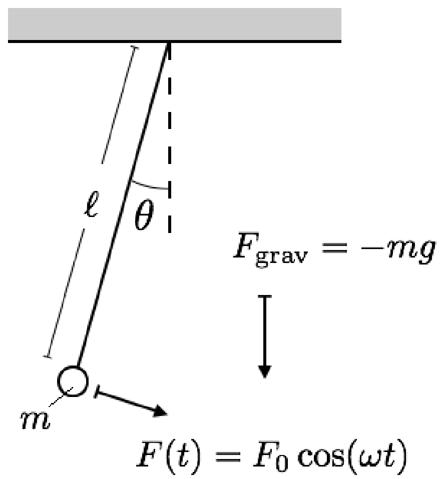

Figure 2: Driven pendulum

Assume throughout that the small angle approximation is valid.

(a) (5 points) What is the equation of motion of the system?

(b) (5 points) At what angular frequency would we need to apply the driving force in order for the system to be in resonance?

(c) (10 points) The mass begins at rest from the position $\theta = 0$. We apply the force with ω equal to the resonance frequency, with the system obeying the initial conditions stated in the prompt. Moreover, say that we can only apply the small-angle approximation when the amplitude of the motion θ_{amp} satisfies $\theta_{\text{amp}} < \frac{3}{4}$ radians. How much time would it take for the small angle approximation to not be a valid description of the system?

Solution:

(a) The equation of motion of the system can be found through energy conservation. For energy conservation, we know that the time derivative of the mechanical energy is power, and power is equal to the external force multiplied by the velocity. Thus

$$\frac{dE_{\text{tot}}}{dt} = vF_{\text{ext}} = \ell\dot{\theta}F_0\cos(\omega t). \tag{35}$$

The total mechanical energy of the system is

$$E_{\text{tot}} = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta). \tag{36}$$

Computing the time-derivative of this result gives us

$$\frac{dE_{\text{tot}}}{dt} = m\ell^2\ddot{\theta}\dot{\theta} + mg\ell\sin\theta\dot{\theta}. \tag{37}$$

Equating this result to Eq.(35) and dividing by $m\ell^2\dot{\theta}$, we find the equation of motion

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = \frac{F_0}{m\ell}\cos(\omega t). \tag{38}$$

Taking the small angle approximation gives us the desired result

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta = \frac{F_0}{m\ell}\cos(\omega t).} \tag{39}$$

Alternatively, we could have used dimensional arguments to note that $\ell\theta$ has units of distance so we can take $\ell\theta$ to be our coordinate variable x in our basic driven undamped oscillator. Then we would have the equation of motion (with $\omega_0^2 = g/\ell$)

$$m\ell\ddot{\theta} + m\omega_0^2\ell\theta = \frac{F_0}{m}\cos\theta, \tag{40}$$

which is equivalent to Eq.(39)

(b) A system is in resonance when the driving frequency is such that the amplitude of driven oscillation is maximized. For Eq.(39) (and from our analysis of the mass-spring analog of this system), we can infer that the general solution is of the form

$$\theta(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0/m\ell}{\omega_0^2 - \omega^2}\cos(\omega t). \tag{41}$$

The amplitude of the driving force term is maximum when $\omega = \omega_0$. Thus we see that the resonance frequency is

$$\boxed{\omega_{\text{res.}} = \sqrt{\frac{g}{\ell}}.} \tag{42}$$

(c) Given the general solution Eq.(41) and the initial conditions $\theta(t=0) = 0$ and $\dot{\theta}(t=0) = 0$, we have the specific solution (again through analogy to the mass-spring system considered in class)

$$\theta(t) = \frac{F_0/m\ell}{\omega_0^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right], \tag{43}$$

when the system is in resonance we have the specific solution

$$\theta(t) = \frac{F_0}{2m\ell\omega_0} t\sin(\omega_0 t). \tag{44}$$

The amplitude of motion for this system is therefore $\theta_{\text{amp}}(t) = F_0/2m\ell\omega_0$. If the small angle approximation is only valid so long as this amplitude is less than $3/4$ radians, then we find that this solution is only valid until the time when $\theta_{\text{amp}}(t) = 3/4$, namely

$$\boxed{t = \frac{3}{4}\frac{2m\ell\omega_0}{F_0} = \frac{3m\sqrt{g\ell}}{2F_0}.} \tag{45}$$