

Units/Dimensions and Limiting cases

As part of any physics class you spend a lot of time answering questions, deriving physical results, and predicting the consequences of experiments both real and imagined. In short you do a lot of analytical work and thereby obtain equations which supposedly describe reality. But how can you be sure? Well, to be precise, it is impossible (even with apparent experimental confirmation) to be absolutely sure that our physical theories correspond to something truly fundamental about our world, but we can often perform simple logical checks without a laboratory to ensure that the associated results are not obviously wrong.

In these notes we discuss two useful checks in this direction in addition to a more general heuristic for checking one's work. The specific definitions of these checks are given here, but you won't be missing anything if you first look at the example in the next section.

Units/Dimensions Check: Since all physical quantities have units and dimensions¹ any expression which represents a physical quantity must have the same units/dimensions as that quantity. This check uses this fact to ensure that an analytic expression has the correct physical units/dimensions. Taking “[A]” to stand for the units² of A we have,

$$\text{If } f(a_1, \dots, a_N) \text{ represents the physical quantity } F, \text{ then } [F] = [f(a_1, \dots, a_N)] \quad (1)$$

Limiting Case Check: This check rests on the idea that the mathematical result modeling a physical phenomena should have analytical properties which mirror the qualitative properties of the phenomena. Expressed mathematically, if we have an analytic function f (with parameters a_1, \dots, a_N which represents a physical quantity F , then we must have

$$\lim_{a_i \rightarrow A} f(a_1, \dots, a_N) = \text{What we expect for } F \text{ when } a_i = A. \quad (2)$$

If either of these checks fail (i.e., the expected limiting cases are not satisfied or $[F] \neq [f]$), then we know our result is incorrect.

1 Example Application

Both of these techniques are best illustrated through examples. We'll consider one example here but there are many more in chapter 1 of [?] which can be found in the Physics Library on the fourth floor of Jefferson. We begin with the following problem statement:

A time dependent force $F(t) = F_0 e^{-bt}$ acts in the x direction on a mass m . The particle has an initial position x_0 and an initial velocity of \dot{x}_0 . The particle's position $x(t)$ is

$$x(t) = x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (e^{-bt} - 1). \quad (3)$$

1. **Units Check:** Given that F_0 has the units of force, b has the units of s^{-1} , and m has the units

¹Dimensions are either Mass, Length, Time (or Charge). Units refer to quantities like Newton's, Joules, etc.

²In physics, $[A]$ typically stands for the *dimension* of A , but we'll only be considering units here so we appropriate this notation.

of kg, check that $x(t)$ has the correct units.

2. **Limiting Cases Check:** Consider $x(t)$ in the limits of:

(a) $F_0 \rightarrow 0$

(b) $m \rightarrow \infty$

(c) $b \rightarrow \infty$

Do these limits of $x(t)$ yield the expected analytic expressions?

1.1 Units Check

Recall that **units** refer to what is typically known as SI units (quantities like Newtons, Joule's, Watts, etc.) while **dimensions** refer to the fundamental quantities length, mass, and time. Units can be expressed in terms of dimensions so we can check either the units or the dimensions of an equation; both must be consistent across the equality for the equation to make sense.

We will consider units here because that is what we typically use to express the physical nature of quantities. We are trying to check the units of the equation

$$x(t) = x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (e^{-bt} - 1). \quad (4)$$

The equation represents the position of a particle, and position has units of meters so for the left hand side of the equation we have

$$[x(t)] = \text{m}. \quad (5)$$

This unit expression must be matched on the right hand side of the equation. Namely, we must find

$$\left[x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (e^{-bt} - 1) \right] = \text{m} \quad (6)$$

Transcendental functions like e^x have no units, so demonstrating this unit equivalence amounts to showing that the quantities

$$x_0, \quad \dot{x}_0 t, \quad \frac{F_0}{bm} t, \quad \frac{F_0}{b^2 m} \quad (7)$$

have units of meters.

The first quantity represents initial position, so the units are automatically manifest:

$$[x_0] = \text{m}. \quad (8)$$

The next quantity is a velocity multiplied by a time, so we have

$$[\dot{x}_0 t] = [\dot{x}_0] \times [t] = \frac{\text{m}}{\text{s}} \times \text{s} = \text{m}, \quad (9)$$

as we expect. By Newtons' Second Law, force must have the same units as mass times acceleration. Thus the units of F_0 are $\text{kg} \cdot \text{m}/\text{s}^2$. Also, because the argument of the exponential e^{-bt} must not have any units (i.e., transcendental functions must be functions of real numbers alone), the units of b are s^{-1} . The units of the final two terms in Eq.(3) are then

$$[F_0 t / bm] = [F_0] \times [t] \times \left[\frac{1}{b} \right] \times \left[\frac{1}{m} \right] = \text{kg} \frac{\text{m}}{\text{s}^2} \times \text{s} \times \frac{1}{\text{s}^{-1}} \times \frac{1}{\text{kg}} = \text{m} \quad (10)$$

$$\left[F_0/b^2m \right] = \left[F_0 \right] \times \left[\frac{1}{b^2} \right] \times \left[\frac{1}{m} \right] = \text{kg} \frac{\text{m}}{\text{s}^2} \times \frac{1}{\text{s}^{-2}} \times \frac{1}{\text{kg}} = \text{m} \quad (11)$$

In all we find Eq.(3) has the correct units.

1.2 Limiting Cases

To consider the limiting cases we apply Eq.(2) to investigate the various limits listed in the problem statement.

1. $F_0 \rightarrow 0$: If we take $F_0 \rightarrow 0$ then the force applied to the particle goes to zero, and we expect the particle to move with constant velocity in time. Taking this limit for Eq.(3) this expectation is confirmed.

$$\lim_{F_0 \rightarrow 0} x(t) = \lim_{F_0 \rightarrow 0} \left[x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (e^{-bt} - 1) \right] = x_0 + \dot{x}_0 t \quad (12)$$

2. $m \rightarrow \infty$: If we take the particle's mass m to infinity, then we expect a finite force $F(t)$ to not change the particle's velocity. Essentially, the particle would appear to not be acted upon a force at all, and we would again expect the particle to move with constant velocity in time. We indeed find this is the case:

$$\lim_{m \rightarrow \infty} x(t) = \lim_{F_0 \rightarrow 0} \left[x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (e^{-bt} - 1) \right] = x_0 + \dot{x}_0 t \quad (13)$$

3. $b \rightarrow 0$: In the function $F(t) = F_0 e^{-bt}$ the parameter b acts as a time constant for the decaying-in-time force. Specifically $\tau = 1/b$ is the amount of time it takes the force to decay to $1/e$ of its initial value. If $b \rightarrow 0$ then this decay time τ goes to infinity and the force never decays in time. In other words, it is simply a constant force. As a constant force, it gives the particle a constant acceleration and we expect the position as a function of time to have the form

$$x(t) \stackrel{?}{=} x_0 + \dot{x}_0 t + \frac{F_0}{2m} t^2. \quad (14)$$

To check this result we focus on the final two terms in Eq.(3). Taking the limit and making use of the Taylor expansion of the exponential we find

$$\begin{aligned} \lim_{b \rightarrow 0} \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (e^{-bt} - 1) &= \lim_{b \rightarrow 0} \frac{F_0}{bm} t + \frac{F_0}{b^2 m} (-bt + \frac{1}{2} b^2 t^2 + \mathcal{O}(b^3)) \\ &= \lim_{b \rightarrow 0} \frac{F_0}{bm} t - \frac{F_0}{bm} t + \frac{F_0}{m} t \frac{1}{2} t^2 + \mathcal{O}(b) \\ &= \frac{F_0}{2m} t^2, \end{aligned} \quad (15)$$

and so Eq.(14) is established. We note we could have obtained this result without an explicit Taylor expansion by applying L'hospital's rule twice.

As a logical point, although Eq.(3) is indeed correct, whenever we apply a limiting case or dimensions check and find that both checks are consistent, at most what we can say is that our checks prove the result is *not definitely wrong* (instead of proving that the result is *definitely correct*). Namely, it is possible for an expression to pass limiting cases and dimensions check and not be the correct result for a physical system.

The Takeaway: When you obtain a final equation for a problem, check that both the units of the problem and limiting cases are what you expect. Given the generality of these two checks, they can be used in any context which involves mathematical modeling of some physical phenomena.