Assignment 2: Simple Harmonic Oscillator and General Oscillations

Due Wednesday June 21, at 9AM under Rene García's door

Preface: This problem set is meant to provide you practice in using complex exponentials to prove trigonometric identities, and in understanding simple harmonic oscillator systems, numerical solutions to differential equations, and small oscillations.

1. **Practice with complex exponentials**

   Use complex exponentials to find an expression for $\sin(\theta_1 + \theta_2 + \theta_3)$ in terms of the sines and cosines of the individual angles.

2. **Changing a spring**

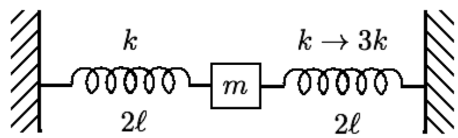


Figure 1

   Two springs each have spring constant k and equilibrium length ℓ. They are both stretched a distance ℓ beyond their equilibrium length and attached to a mass m and two walls, as shown in Fig. 1. At a given instant, the right spring constant is somehow magically changed to $3k$ (the relaxed length remains ℓ). What is the resulting $x(t)$? Take the initial position of the mass to be $x = 0$.

3. **Removing a spring**

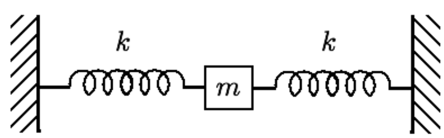
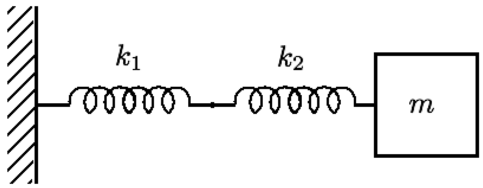


Figure 2

   The springs in Fig. 2 are at their equilibrium length. The mass oscillates along the line of the springs with amplitude d. At the moment (let this be $t = 0$) when the mass is at position $x = d/2$ (and moving to the right), the right spring is removed. What is the resulting $x(t)$? What is the amplitude of the new oscillations?

4. **Effective Spring Constant**



(a)

Figure 3

Two springs with spring constants k_1 and k_2 are connected in series, as shown in Fig. 3a. What is the effective spring constant k_{eff}? In other words, if the mass is displaced by x, find the k_{eff} for which the force equals $F = -k_{\text{eff}} x$.

5. **Oscillation of bead with gravitating masses**
A bead of mass m slides without friction on a smooth rod along the x axis. The rod is equidistant between two spheres of mass M. The spheres are located at $x = 0$, $y = \pm a$ as shown, and attract the bead gravitationally.

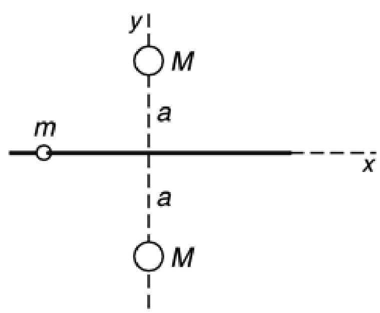


Figure 4

Say the mass begins its "small-oscillation" motion from the origin with an initial velocity $\dot{x}(t = 0) = v_0$. Write the subsequent "small-oscillation" motion as a function of time t, and in terms of G (Newton's Gravitational Constant), M, m, a, and v_0.

(Hint: You should solve this problem in two main steps. First, find the angular frequency of small oscillations. Then use this frequency in the simple harmonic oscillator equation of motion and in the general solution to that equation.)

6. **Numerical Solution to Differential Equations**
Euler's method is a numerical procedure for solving differential equations which makes use of the first order changes in functions. Namely for a time ϵ which is small relative to the larger time scales we are interested in, we can make the approximation

$$x(t + \epsilon) = x(t) + \epsilon\, \dot{x}(t) + \mathcal{O}(\epsilon^2) \tag{1}$$

$$\dot{x}(t + \epsilon) = \dot{x}(t) + \epsilon\, \ddot{x}(t) + \mathcal{O}(\epsilon^2), \tag{2}$$

where $\mathcal{O}(\epsilon^2)$ stands for terms of order ϵ^2 or higher. Therefore, if we have an equation of motion for $x(t)$ which is written as

$$\ddot{x}(t) = F(x, \dot{x}, t), \tag{3}$$

where F is a function of position x, velocity $\dot{x}$, and/or time t, we can solve for $x(t)$ step-by-step. Assuming we know $x(t)$ and $\dot{x}(t)$, at $t = 0$ we can use Eq.(1) to find x at $t = 0 + \epsilon$ and we can use Eq.(2) and Eq.(3) to find $\dot{x}$ at $t = 0 + \epsilon$. In this way, we can find $x(t)$ from $t = 0$ to an arbitrary time t by moving in steps of ϵ. In this problem we implement this procedure to solve the equation of motion of a pendulum

Before we can implement this code we must get set up with our numerical program *Mathematica*. Here are the preliminary steps before you can begin this problem

(i) Log in to your account in MIT's Athena Cluster, and go to the course website.

(ii) Download the code "numerical_diff_eq.nb" from the course webpage and open it in *Mathematica*.

(iii) Select a block of code and run it by pressing `Shift+Enter`.

Now we can begin the problem.

(a) For each line of the code, write a sentence explaining the line's utility in the overall code. (You can annotate the code itself)

(b) A student wrote this code intending to produce a plot of simple harmonic motion, but he made a few errors. Given Eq.(1), Eq.(2), and the simple harmonic equation of motion

$$\ddot{x} = -\omega_0^2 x \tag{4}$$

modify the code to plot $x(t)$ as a function of time with the initial conditions $x(t = 0) = 1.0$ m, $\dot{x}(t = 0) = 1.0$ m/s^2, and with $\omega_0^2 = 5.0$ rad/s^2. (*Hint: The simplest way to do this is to just copy and paste the incorrect code and make modifications to make it correct.*)

(c) **Extra Credit:** Taking $\theta(t = 0) = \pi/2$ rad, $\dot{\theta}(t = 0) = 0$, $g = -9.8$ m/s^2, modify the Euler's algorithm code to plot $\theta(t)$ as a function of time for the pendulum equation of motion

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta. \tag{5}$$

Useful Information: The *Mathematica* function for $\sin(x)$ is "`Sin[x]`". What is the period of the pendulum motion in the plot? (i.e., how much time does it take to go from one amplitude to another?). How does this compare to the prediction given by $T = 2\pi\sqrt{\ell/g}$?

Submitting: As your submission for this part of the assignment, you can print out the entire *Mathematica* notebook which should include your annotations of the existing code, and your plots of simple harmonic and pendulum motion (in addition to the answers for the questions in (c)).

7. **Small Oscillations about equilibria**

A particle of mass m moves on the x axis with potential energy

$$U(x) = \frac{E_0}{a^4}\left(x^4 + 4ax^3 - 8a^2x^2\right). \tag{6}$$

Write out **two** precisely stated physics questions we could ask about the oscillatory behavior of a particle in this system. Answer both of your questions using your understanding of near-equilibrium oscillations. *(Hint: This potential may look complicated, but near certain values of x, it can be approximated as something much simpler.)*