Assignment 4: Coupled Oscillations and Fourier Series

Due Wednesday July 5, at 9AM under Rene García's door

Preface: In this assignment, we practice solving differential equations, studying the nonlinear properties of the pendulum. practice computing the properties of coupled oscillator systems, and use Fourier Series to derive an identity for π^2.

1. **Third-order differential equation**
   Find three independent solutions to the third-order differential equation

$$\frac{d^3}{dt^3}x(t) + x(t) = 0. \tag{1}$$

   You can write these solutions in terms of complex exponentials. But if $x(t)$ is a real quantity what is the general solution to this differential equation (*Your differential equation should have three undetermined constants because this is a third order differential equation.*)

2. **Corrections to Pendulum Period**
   We previously showed, that in the small angle approximation the period of a pendulum is $T = 2\pi\sqrt{\ell/g}$. In this problem we compute corrections to this result to show that the period is actually dependent on the amplitude of our motion.

   (a) Using the equation for energy conservation of a pendulum and the heuristic $dt = d\theta/\dot{\theta}$, show that the exact expression for the period of a pendulum (which begins from rest at a angle θ_0) is

$$T = \sqrt{\frac{8\ell}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \tag{2}$$

   (b) We will find an expression for T up to second order in θ_0 in the following way. First, use the identity $\cos\phi = 1 - 2\sin^2(\phi/2)$ to write T only in terms of sines. This way we have quantities which go to zero as $\theta \to 0$. Next, make the change of variables $\sin x = \sin(\theta/2)/\sin(\theta_0/2)$. Finally, using the binomial series, expand the integrand in powers of θ_0 and perform the first two non-zero integrals to show that

$$T \simeq 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right) \tag{3}$$

3. **Two coupled oscillators**
   We have two masses m_1 and m_2 which are free to move along the horizontal axis. The masses have a spring of spring constant k joining them.

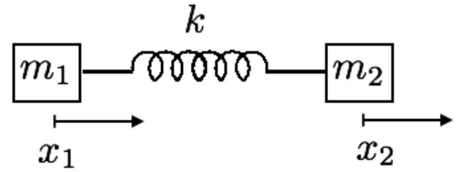

Figure 1

We define the center of mass of the particles as

$$X_{\text{cm}} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{4}$$

and the distance between them as

$$R \equiv x_2 - x_1 \tag{5}$$

Given the equations of motion of the system, what is the most general solution for $x_1(t)$ and $x_2(t)$? It will be helpful to define results in terms of the **reduced mass** μ of the system:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{6}$$

(Hint: I suggest you begin by use the "add and subtract" method outlined at the beginning of Lecture notes 06 and then use Eq.(4) *and Eq.*(5) *to find equations of motion for X_{cm} and R).*

4. **Three coupled masses**
We have three identical masses m joined by three identical springs of spring constant k. The masses are constrained to move on a line.

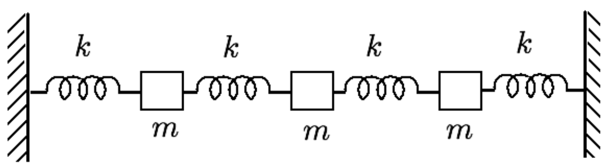

Figure 2

(a) What are the normal mode frequencies and corresponding normal modes of this system?

(b) What is the general solution for x_1, x_2, and x_3 for this system? (After finding the normal modes and normal mode frequencies you can simply insert your results into the general solution Eq. (37) from Lecture Notes 06.)

5. **The Strogatz Sync**
Watch the video in the link➡ from 11:30 to 14:10 (Feel free to watch the entire video for some context, but doing so is not necessary for this problem). Our goal in this problem is to develop a quantitative explanation for what Strogatz observes in his "two-metronomes-on-a-platform on-two-bottles" system. In particular we want to understand why the two metronomes evolve to be in sync.

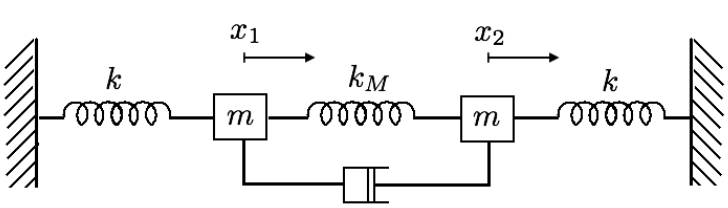

Figure 3

To this end, we depict the Strogatz's system as the system in Fig. 3. We will ignore the motion of the platform and the bottle and only focus on the motion of the metronomes. We model each metronome as a mass m coupled to a wall by a spring of spring constant k. Without the platform between them, the two metronomes oscillate independently of one another. Similarly, without anything coupling the two masses in our model they also oscillate independently of one another. Thus we model the platform in the video by including a spring of spring constant k_M between the two masses and a dashpot connecting them.

The dashpot exerts a force bv on the masses, where v is the relative velocity of the two ends of the dashpot. The dashpot force opposes motion. Let x_1 and x_2 be the displacements of the two masses from equilibrium.

(a) Using the above model of Strogatz's system, explain (as quantitatively as possible) why the two metronomes eventually sync up. In other words show why the antisymmetric mode (the "$x_1 = -x_2$" mode) decays away to zero, and why only the symmetric mode (the "$x_1 = x_2$" mode) remains. Also, use the model you develop to approximately calculate how much time it would take the metronomes to sync if they began from arbitrary positions and initial velocities.

6. **Fourier Series**

We previously found that the general solution to the wave equation for a string fixed at the ends of a domain $[0, L]$ is

$$y(x,t) = \sum_{n=1}^{\infty} \left[\alpha_n \sin(\omega_n t) + \beta_n \cos(\omega_n t)\right] \sin\left(\frac{n\pi}{L} x\right), \tag{7}$$

where $\omega_n = n\pi v/L$, where v is the velocity of the wave.

(a) At $t = 0$, we pluck a string so that it has the form.

$$y(x,0) = \begin{cases} x & \text{for } 0 \leq x \leq L/2 \\ L - x & \text{for } L/2 \leq x \leq L \end{cases} \tag{8}$$

At this initial time, the string is at rest. Determine the values of the α_n and β_n coefficients in Eq.(7). *(One of these calculations is really easy. The other will require integration by parts.)*

(b) **Extra Credit–Computing π^2:** Using Eq. (57) in Lecture notes 07, and the above results we can compute the total energy (for $t > 0$) of the string as a series. Using Eq. (54) we can compute the total energy at $t = 0$. By conservation of energy, these two energies must be equal. Use this fact to derive the identity

$$\frac{\pi^2}{8} = \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots, \tag{9}$$

where the sum runs over all odd integers.