Assignment 5: Fourier Series and Wave Equations

Due Wednesday July 12, at 9AM under Rene García's door

Preface: In this assignment, we build a better understanding of Fourier Series and derive various wave equations.

1. **Fourier Series identities**
   Using the identities

$$\cos\theta\cos\phi = \frac{1}{2}\left[\cos(\theta-\phi)+\cos(\theta+\phi)\right] \quad \text{and} \quad \sin\theta\cos\phi = \frac{1}{2}\left[\sin(\theta-\phi)+\sin(\theta+\phi)\right], \tag{1}$$

   Compute the values of

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx \tag{2}$$

   and

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx \tag{3}$$

   in terms of the Kronecker delta δ_{nm}. (You might be able to guess the answer, but you also need to derive it.)

2. **Summing Fourier Series**
   In the previous assignment, we found that the function

$$y(x,0) = \begin{cases} x & \text{for } 0 \le x \le L/2 \\ L-x & \text{for } L/2 \le x \le L \end{cases} \tag{4}$$

   could be expressed as the Fourier series

$$y(x,0) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}x\right). \tag{5}$$

   Download the notebook "fourier_series.nb" from the website. The notebook as it is generates a Taylor Series approximation to $\sin(x)$.

   (a) Run the first part of the code, and then annotate the subsequent lines which generate the Taylor Series approximation to $\sin(x)$. Your annotation to explain the meaning of each line of code.

   (b) Run the second part of the code, and using the format provided by the first part of the code, write code which computes a Fourier Series approximation of Eq.(4). (We will take $L = 1$ for simplicity.)

   (c) Using the written code, plot three approximations of Eq.(4). Namely, approximate $y(x,0)$ by
      i. the first five non-zero terms in Eq.(5)
      ii. the first 10 non-zero terms in Eq.(5)
      iii. the first 25 non-zero terms in Eq.(5)

      and plot each of the three approximations.

   (d) Comment on what plot you expect to see when you add more and more terms to your summation.

3. **Fourier Series with new Boundary Conditions**
In discussing Fourier Series solutions to the wave equation, we derived Eq. 35 of Lecture notes 07 by imposed the boundary conditions

$$y(x=0,t)=0, \qquad y(x=L,t)=0, \tag{6}$$

on our string.

For a different string system, one with **free ends**, the boundary conditions would be

$$\frac{\partial}{\partial x}y(x=0,t)=0, \qquad \frac{\partial}{\partial x}y(x=L,t)=0. \tag{7}$$

(a) Show that the general solution to the wave equation for these boundary conditions is

$$y(x,t)=\frac{\alpha_0}{2}+\sum_{k=1}^{\infty}\left[\alpha_n\cos(\omega_n t)+\beta_n\sin(\omega_n t)\right]\cos\left(\frac{n\pi}{L}x\right) \tag{8}$$

(b) If we are given the initial conditions $y(x,0)$ and $\dot{y}(x,0)$ what are α_n and β_n? *Hint: You should use one of the results from Problem 1 of this assignment.*

(c) Using the above derived formulas, determine α_n and β_n for a string which begins at rest in the position

$$y(x,0)=\begin{cases}\dfrac{L}{2} & \text{for } 0\le x<L/2\\[2ex] -\dfrac{L}{2} & \text{for } L/2\le x\le L\end{cases} \tag{9}$$

(d) **Extra Credit:** By equating Eq.(9) and Eq.(8) (with α_n and β_n determined), and setting $x=0$, prove the identity

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n} \tag{10}$$

4. **String Wave Equation with Gravity**
Review the derivation of the "Wave Equation for Transverse Waves" in the Lecture 07 notes. In the associated system, the masses were moving vertically but we did not consider them under the influence of gravity.For this problem, we will assume their is a constant gravitational field in the system.

(a) Using the derivation in the lecture notes as a model, derive the following wave equation for a string in a constant gravitational field:

$$\mu\frac{\partial^2}{\partial t^2}y(x,t)=T\frac{\partial^2}{\partial x^2}y(x,t)-\mu g. \tag{11}$$

(b) Consider our string in the presence of gravity to be at rest. Given the boundary conditions,

$$y(x=0,t)=0, \qquad y(x=L,t)=0, \tag{12}$$

what is $y(x)$, the function the rope makes in space?

(c) Fig. 1 depicts the system in (b) at rest. Determine the value of θ in terms of the parameters of the system. (*Hint: You should first determine the relationship between the slope of the string and the angle θ.*)

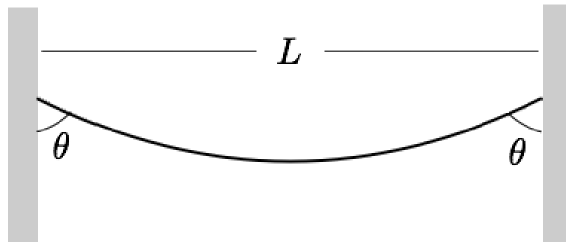

Figure 1: String in a gravitational field

5. **Nonlinear Wave Equation**
Review the derivation of the "Wave Equation for Transverse Waves" in the Lecture 07 notes. In completing the derivation of the wave equation, we took the "strong coupling" limit $\ell_R \ll a$ (i.e., the rest length of the spring is much less than the spacing between the masses). Here we will take the weak coupling limit where
$$\ell_R = a, \tag{13}$$
(i.e., the rest length of the spring is the same as the spacing between the masses). We will also take $|y_j - y_{j-1}| \ll a$, for all j.

   (a) Under the above listed conditions, and working through a calculation similar to that in the notes, derive the nonlinear wave equation
$$\mu \frac{\partial^2 y}{\partial t^2} = \frac{T}{2} \frac{\partial}{\partial x} \left[\left(\frac{\partial y}{\partial x} \right)^3 \right]. \tag{14}$$

   (b) Does the solution $y(x,t) = Ae^{i(kx-\omega t)}$ solve Eq.(14)? Would any sinusoidal solution solve it? (Explain why not)