Solutions 1: Introduction and Kinematics Review

Due Wednesday June 14 at 9AM

1. **(10 pts) Introduce Yourself**

   Full score if complete.

2. **Power Series and Perturbation Theory**

   (a) (5 pts) We want to expand the equation

$$\alpha x^2 - \beta = \varepsilon x^3, \tag{1}$$

Given the power series

$$x = x_{(0)} + \varepsilon\, x_{(1)} + \varepsilon^2\, x_{(2)} + \cdots . \tag{2}$$

up to second order in ε. Considering the left-hand side (LHS) of the given equation and the assumed solution in terms of $x^{(k)}$ (for $k \geq 0$), we find

$$\begin{aligned} \alpha x^2 - \beta &= \alpha \left(x_{(0)} + \varepsilon\, x_{(1)} + \varepsilon^2\, x_{(2)} + \cdots\right)^2 - \beta \\ &= \alpha x_{(0)}^2 - \beta + \alpha\varepsilon^2\, x_{(1)}^2 + 2\varepsilon\, x_{(1)} x_{(0)} + 2\alpha\varepsilon^2\, x_{(2)} x_{(0)} + \cdots \\ &= \alpha x_{(0)}^2 - \beta + 2\alpha\varepsilon\, x_{(1)} x_{(0)} + \varepsilon^2 \left(\alpha x_{(1)}^2 + 2\alpha x_{(2)} x_{(0)}\right) + \cdots \end{aligned} \tag{3}$$

where $\cdots$ stands in for terms of order ε^3 or higher. Similarly considering the right-hand side (RHS), we have

$$\begin{aligned} \varepsilon x^3 &= \varepsilon \left(x_{(0)} + \varepsilon\, x_{(1)} + \varepsilon^2\, x_{(2)} + \cdots\right)^3 \\ &= \varepsilon\, x_{(0)}^3 + 3\varepsilon^2\, x_{(0)}^2 x_{(1)} + \cdots , \end{aligned} \tag{4}$$

where $\cdots$ stands again in for terms of order ε^3 or higher. Thus we find that the defining equation, expanded as a power series in ε (up to second order) is

$$\alpha x_{(0)}^2 - \beta + 2\alpha\varepsilon\, x_{(1)} x_{(0)} + \varepsilon^2 \left(\alpha\, x_{(1)}^2 + 2\alpha x_{(2)} x_{(0)}\right) + \cdots = \varepsilon\, x_{(0)}^3 + 3\varepsilon^2\, x_{(0)}^2 x_{(1)} + \cdots . \tag{5}$$

   (b) (5 pts) In the next step of the perturbation theory procedure, we match the coefficients of each order of ε across the equal sign in Eq.(5). That is we say the coefficient of the ε-independent term on the LHS of Eq.(5) is equivalent to the ε-independent term on the RHS of Eq.(5); the linear-in-ε term on the LHS of Eq.(5) is equivalent to the coefficient of the linear-in-ε term on the RHS of Eq.(5) and so on. Doing so, we find the system of equations,

$$\begin{aligned} \alpha x_{(0)}^2 - \beta &= 0 \\ 2\alpha\varepsilon\, x_{(1)} x_{(0)} &= \varepsilon\, x_{(0)}^3 \\ \alpha\varepsilon^2\, x_{(1)}^2 + 2\alpha\varepsilon^2\, x_{(2)} x_{(0)} &= 3\varepsilon^2\, x_{(0)}^2 x_{(1)} \end{aligned} \tag{6}$$

or, with the ε powers divided out

$$\alpha x_{(0)}^2 - \beta = 0 \tag{7}$$

$$2\alpha\, x_{(1)} x_{(0)} = x_{(0)}^3 \tag{8}$$

$$\alpha\left(x_{(1)}^2 + 2x_{(2)}x_{(0)}\right) = 3x_{(0)}^2 x_{(1)} \tag{9}$$

Solving Eq.(7), we find the two solutions

$$x_{(0),\pm} = \pm\left(\frac{\beta}{\alpha}\right)^{1/2}. \tag{10}$$

Using this solution in Eq.(8), gives us

$$x_{(1)} = \frac{1}{2\alpha}x_{(0)}^2 = \frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right). \tag{11}$$

And using Eq.(10) and Eq.(11) in Eq.(9) gives us

$$\begin{aligned} x_{(2),\pm} &= \frac{1}{2x_{(0)}}\left(\frac{3}{\alpha}x_{(0)}^2 x_{(1)} - x_{(1)}^2\right) \\ &= \pm\frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{1/2}\left(\frac{3}{\alpha}\cdot\left(\frac{\beta}{\alpha}\right)\cdot\frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right) - \frac{1}{4\alpha^2}\left(\frac{\beta}{\alpha}\right)^2\right) \\ &= \pm\frac{5}{8\alpha^2}\left(\frac{\beta}{\alpha}\right)^{3/2}. \end{aligned} \tag{12}$$

(c) (5 pts) By the definition of power series in ε given in the prompt, we have that the solution to

$$\alpha x^2 - \beta = \varepsilon x^3 \tag{13}$$

can be written as

$$x = x_{(0)} + \varepsilon\, x_{(1)} + \varepsilon^2\, x_{(2)} + \cdots. \tag{14}$$

Given Eq.(10), Eq.(11), and Eq.(12), we find that Eq.(14) becomes

$$x_{\pm} = \pm\left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\,\frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right) \pm \varepsilon^2\frac{5}{8\alpha^2}\left(\frac{\beta}{\alpha}\right)^{3/2} + \cdots \tag{15}$$

and the two approximate solutions to Eq.(13) are

$$x_{+} = \left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\,\frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right) + \varepsilon^2\,\frac{5}{8\alpha^2}\left(\frac{\beta}{\alpha}\right)^{3/2} + \cdots \tag{16}$$

and

$$x_{-} = -\left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\,\frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right) - \varepsilon^2\,\frac{5}{8\alpha^2}\left(\frac{\beta}{\alpha}\right)^{3/2} + \cdots, \tag{17}$$

If we take $\varepsilon \to 0$ in Eq.(16) and Eq.(17), we find $x_+ = +\sqrt{\beta/\alpha}$ and $x_- = -\sqrt{\beta/\alpha}$ as we should expect for the solutions of Eq.(13) when $\varepsilon \to 0$.

(d) (5 pts) We want to find an approximate solution to the equation

$$x^2 - 1 = 0.1x^3 \tag{18}$$

Eq.(18) is identical to Eq.(13) with $\alpha = \beta = 1.0$ and $\varepsilon = 0.1$. Therefore we can use Eq.(16) and

Eq.(17) to approximately solve Eq.(18). Doing so, we find

$$\begin{aligned} x_+ &= \left(\frac{1.0}{1.0}\right)^{1/2} + (0.1)\,\frac{1}{2(1.0)}\left(\frac{1.0}{1.0}\right) + (0.1)^2\,\frac{5}{8(1.0)^2}\left(\frac{1.0}{1.0}\right)^{3/2} + \cdots \\ &= 1.0 + 0.1/2 + 0.01\times(0.625) + \cdots \\ &= 1.056 + \cdots\,, \end{aligned} \tag{19}$$

and

$$\begin{aligned} x_- &= -\left(\frac{1.0}{1.0}\right)^{1/2} + (0.1)\,\frac{1}{2(1.0)}\left(\frac{1.0}{1.0}\right) - (0.1)^2\,\frac{5}{8(1.0)^2}\left(\frac{1.0}{1.0}\right)^{3/2} + \cdots \\ &= -1.0 + 0.1/2 - 0.01\times(0.625) + \cdots \\ &= -0.956 + \cdots\,, \end{aligned} \tag{20}$$

where $\cdots$ stand for $\varepsilon^3 = 0.001$ order corrections to this result. Solving Eq.(**??**) with *Wolframalpha*, we find that the two real solutions are

$$x_+ \approx 1.057 \quad \text{and} \quad x_- \approx -0.955 \quad \text{[Numerical Results]}, \tag{21}$$

both of which are in good agreement with Eq.(19) and Eq.(20), respectively.

We note finally that the error between the approximate solution and *Wolframalpha*'s result is on the order of 0.001. This makes sense given the ε^3 terms we neglected in the perturbation series: For $\varepsilon = 0.1$ we have $\varepsilon^3 = 0.001$, so our approximate values Eq.(16) and Eq.(17) should differ from the true result by $\varepsilon^3 = 0.001$.

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3. **(10 pts) Marble and elevator**

We want to find the height of the elevator given the kinematical properties of the marble's motion. We are not given the speed of the elevator, so we must either solve for it or eliminate it from our equations.

If, starting from $t = 0$, the elevator ascends uniformly until a time $t = T_1$, then at this time the elevator would be at the height

$$h = v_0 T_1, \tag{22}$$

where v_0 is the speed of the elevator's ascension. When the marble is released from the elevator, it has the elevator's vertical velocity of $+v_0$ and it is at the height h. After a time $T_2 - T_1$, the marble is on the ground (i.e., at zero height). With this information, the kinematics of the marble at the point of zero height is given by

$$0 = h + v_0(T_2 - T_1) - \frac{1}{2}g(T_2 - T_1)^2, \tag{23}$$

where $T_2 - T_1$ (with $T_2 \geq T_1$) is the time the ball takes to fall to the ground. Using Eq.(22) to solve for the unknown variable v_0, we find $v_0 = h/T_1$. Inserting this result into Eq.(23), yields

$$0 = h + \frac{h}{T_1}(T_2 - T_1) - \frac{1}{2}g(T_2 - T_1)^2 = h\frac{T_2}{T_1} - \frac{1}{2}g(T_2 - T_1)^2. \tag{24}$$

Therefore, the value of h is

$$h = \frac{gT_1}{2T_2}(T_2 - T_1)^2. \tag{25}$$

We note that the units of Eq.(25) are $[g] \times [T_1/T_2] \times [(T_2 - T_1)^2] = \text{m/s}^2 \times \text{s/s} \times \text{s}^2 = \text{m}$ as we should expect for a height.

Checking a limiting case for Eq.(25) we note that as $T_1 \to 0$, $h \to 0$ (i.e., if we release the marble immediate after the elevator starts to rise, it doesn't get too far off the ground). Similarly, we find $T_2 \to T_1$, gives us $h \to 0$ meaning if the marble takes the same amount of time to fall as to it took the elevator to rise, then the marble hasn't risen at all.

Finally checking the answer quoted in the text, we find, with $T_1 = 4$ sec and $T_2 = 8$ sec (and being on earth),

$$h = \frac{gT_1}{2T_2}(T_2 - T_1)^2 = 9.8\text{ m/s}^2 \times \frac{4\text{ sec}}{2(8\text{ sec})}(4\text{ sec})^2 = 39.2\text{ m}. \tag{26}$$

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4. **(10 pts) Thought Experiments in Kinematics**

A ball is thrown off a cliff at a height h and at an angle such that the ball covers a maximum horizontal distance. We want to find the value of d_{max} given the possibilities listed in the prompt. We will proceed through each possible answer, explaining why the answer contradicts (or does not contradict) simple properties we expect the motion should have.

–
$$d_{\text{max}} \stackrel{?}{=} \frac{gh^2}{v^2} \tag{27}$$

Incorrect answer. As $v \to 0$, this expression becomes huge which contradicts what we know *should* happen. If the speed of the ball went to zero, the height should go to zero as well.

–
$$d_{\text{max}} \stackrel{?}{=} \frac{v^2}{g} \tag{28}$$

Incorrect answer. This result is independent of h which contradicts what we know *should* happen. The maximum distance when $h = 0$ should not be the same as the maximum distance when $h \neq 0$, because the ball does not stay in the air for the same amount of time between the two cases, and consequently the trajectory has a different y vs x behavior and a different maximum distance.

–
$$d_{\text{max}} \stackrel{?}{=} \sqrt{\frac{v^2 h}{g}} \tag{29}$$

Incorrect answer. This result goes to zero as $h \to 0$ which contradicts what we know *should* happen. The maximum distance when $h = 0$ is not equal to zero.

–
$$d_{\text{max}} \stackrel{?}{=} \frac{v^2}{g}\sqrt{1 + \frac{2gh}{v^2}} \tag{30}$$

Correct answer. This result is consistent with all the limiting cases we expect from intuition. If $h \to 0$, we find the maximum distance is v^2/g (which is the correct answer for the maximum range of the standard projectile motion. See "Range of Projectile Motion➡). As $v \to 0$, we have $2gh/v^2 \gg 1$ and the result approximates to $\simeq (v^2/g)\sqrt{2gh/v^2} = v\sqrt{2h/g}$ which goes to 0 as v goes to zero; thus as the velocity goes to zero so does the maximum height as we expect. Moreover this result is exclusively positive for valid ranges of g, h, and v^2. If $-2gh > v^2$, then there is no maximum distance.

–
$$d_{\text{max}} \stackrel{?}{=} \frac{v^2}{g}\left(1 + \frac{2gh}{v^2}\right) \tag{31}$$

Incorrect answer. This result remains non-zero as $v \to 0$ which contradicts what we know *should* happen. As the speed of the ball goes to zero, the maximum distance we can achieve should also go to zero because a stationary ball doesn't move horizontally (or in any direction). But as $v \to 0$, the above expression reduces to $2h$, twice the height we're throwing from which does not make sense.

–

$$d_{\text{max}} \stackrel{?}{=} \frac{v^2/g}{1 - 2gh/v^2} \tag{32}$$

Incorrect answer. This result has a **singularity** (i.e., $d_{\text{max}} \to \infty$) at $v^2 = 2gh$ which contradicts what we know *should* happen. There is no height where we should expect the maximum distance to go to infinity.

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5. **(10 pts) Projectile Motion on a Hill**

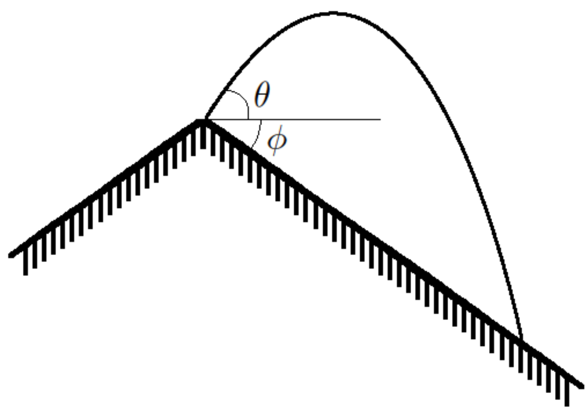

Figure 1: Projectile Motion

The purpose of this problem is to get you to practice asking questions about physical systems and formulating these questions in a way that they are answerable using the physics you've learned. This is an open ended problem, so there are many questions you could ask. Some of them are:

- How far along the decline does the rock travel?
- Assuming ϕ is fixed, what angle θ maximizes the total distance along the decline the rock travels?
- How far (vertically) below its starting point does the rock land?
- What is the potential energy of the rock at the lowest point in its trajectory?
- What is the total time it takes the rock to complete its trajectory?
- What angle maximizes the area under the curve of the trajectory?
- How would these results change if $\phi < 0$ (i.e., there was a incline instead of a decline).

We will answer the first two, namely we will determine the total distance along the decline the ball travels and determine the angle θ which maximizes this distance given a fixed ϕ.

For this problem the kinematical equations for x and y have their standard form

$$x(t) = (v_0 \cos\theta)t \qquad y(t) = (v_0 \sin\theta)t - \frac{1}{2}gt^2. \tag{33}$$

If we take ℓ to be the total distance the object travels along the decline, then by geometry it has traveled a total horizontal distance $\ell \cos\phi$ and fallen a distance $\ell \sin\phi$ from its starting point. Assuming it completed its trajectory in a time t_f, we have the equations

$$x(t_f) = \ell \cos\phi = v_0 \cos\theta t_f \tag{34}$$

$$y(t_f) = -\ell \sin\phi = v_0 \sin\theta t_f - \frac{1}{2} g t_f^2 \tag{35}$$

Solving Eq.(34) for t_f, we find

$$t_f = \frac{\ell \cos\phi}{v_0 \cos\theta}, \tag{36}$$

and plugging this result back into Eq.(35) gives us

$$-\ell \sin\phi = \ell \cos\phi \tan\theta - \frac{g \ell^2 \cos^2\phi}{2 v_0^2 \cos^2\theta}. \tag{37}$$

Dividing by ℓ to eliminate the extraneous $\ell = 0$ solution we find

$$-\sin\phi = \cos\phi \tan\theta - \frac{g \ell \cos^2\phi}{2 v_0^2 \cos^2\theta} \tag{38}$$

which when solved for ℓ gives

$$\ell = \frac{2 v_0^2}{g} \frac{\cos^2\theta \left(\tan\phi + \tan\theta\right)}{\cos\phi}, \tag{39}$$

which is the total distance the rock travels along the decline. We note that, as we expect, $\phi = 0$ reduces to the standard $2 v_0^2 \cos\theta \sin\theta / g$ result.

To compute the angle θ where this distance is maximum we differentiate Eq.(39) with respect to theta to find

$$\begin{aligned}
\frac{d\ell}{d\theta} &= \frac{2 v_0^2}{g \cos\phi} \Big[-2 \cos\theta \sin\theta \left(\tan\phi + \tan\theta\right) + \cos^2\theta \left(\sec^2\theta\right) \Big] \\
&= \frac{2 v_0^2}{g \cos\phi} \Big[-2 \cos\theta \sin\theta \tan\phi - 2 \sin^2\theta + 1 \Big] \\
&= \frac{2 v_0^2}{g \cos\phi} \Big[-\sin 2\theta \tan\phi + \cos 2\theta \Big],
\end{aligned} \tag{40}$$

which implies Eq.(39) has a critical point at θ given by

$$\cot 2\theta_0 = \tan\phi. \tag{41}$$

Differentiating Eq.(40) once again and setting $\theta = \theta_0$, we find

$$\left.\frac{d^2\ell}{d\theta^2}\right|_{\theta=\theta_0} = -\frac{4 v_0^2}{g \cos\phi} \Big[\cos 2\theta_0 \tan\phi + \sin 2\theta_0 \Big] < 0. \tag{42}$$

Thus with Eq.(**??**), we find that the angle

$$\theta = \frac{1}{2} \tan^{-1}(\cot\phi) \tag{43}$$

yields a trajectory has a maximum travel along the decline.

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