Solutions 2: Simple Harmonic Oscillator and General Oscillations

Due Wednesday June 21, at 9AM under Rene García's door

Preface: This problem set is meant to provide you practice in using complex exponentials to prove trigonometric identities, and in understanding simple harmonic oscillator systems, numerical solutions to differential equations, and small oscillations.

1. **(5 points) Practice with complex exponentials**
We want to express $\sin(\theta_1+\theta_2+\theta_3)$ as a sum of products of the sines and cosines of the individual angles. To do so we will begin by expressing the given quantity as the imaginary part of a product of exponentials. Doing so we find

$$\sin(\theta_1+\theta_2+\theta_3) = \text{Im}\left[e^{i(\theta_1+\theta_2+\theta_3)}\right] = \text{Im}\left[e^{i\theta_1}e^{i\theta_2}e^{i\theta_3}\right]. \tag{1}$$

Expanding the right hand side of this equation, we find

$$\begin{aligned}
&\sin(\theta_1+\theta_2+\theta_3) \\
&\quad = \text{Im}\Big[(\cos(\theta_1)+i\sin(\theta_1))(\cos(\theta_2)+i\sin(\theta_2))(\cos(\theta_3)+i\sin(\theta_3))\Big] \\
&\quad = \text{Im}\Big[\cos\theta_1\cos\theta_2\cos\theta_3 + i\cos\theta_1\cos\theta_2\cos\theta_3 + i\cos\theta_1\sin\theta_2\cos\theta_3 \\
&\qquad\quad - \cos\theta_1\sin\theta_2\sin\theta_3 + i\sin\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 \\
&\qquad\quad - \sin\theta_1\cos\theta_2\sin\theta_3 - i\sin\theta_1\sin\theta_2\sin\theta_3\Big].
\end{aligned} \tag{2}$$

And taking the imaginary part gives us

$$\sin(\theta_1+\theta_2+\theta_3) = \cos\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3, \tag{3}$$

a result which could also be derived by using the sum of angle formulas for the sine function.

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2. **(10 points) Changing a spring**

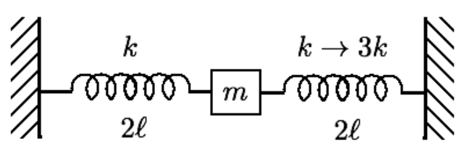

Figure 1

We want to determine the dynamics of the mass m after we change the spring constant of the right spring from k to $3k$. Before we make the change, the dynamics of x is governed by the following equation of motion:

$$m\ddot{x} = F_{\text{net}} = -k(x+\ell) - k(x-\ell) = -2kx. \tag{4}$$

The first term after the first equality in Eq.(4) arises from the the fact that the left spring is at equilibrium when $x=-\ell$ (i.e., when the mass is a distance ℓ from the left wall), and the second term arises from the fact that the right spring is at equilibrium when $x = +\ell$ (i.e., when the mass is a distance ℓ from the right wall). Right when the spring is changed, the mass is at rest at the equilibrium position ($x = 0$ by Eq.(4)) given by these two spring forces, so we have the initial conditions

$$x(t = 0) = 0, \qquad \dot{x}(t = 0) = 0. \tag{5}$$

After we change the spring constant of the right spring, the new equation of motion is

$$m\ddot{x} = F_{\text{net}} = -k(x + \ell) - 3k(x - \ell) = -4k(x - \ell/2), \tag{6}$$

indicating that the new equilibrium position is at $x = \ell/2$. Defining $X \equiv x - \ell/2$ and

$$\omega_0^2 = 4k/m,$$

Eq.(6) reduces to the equation of motion

$$\ddot{X} + \omega_0^2 X = 0 \tag{7}$$

Which has the general solution

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t). \tag{8}$$

Writing this in terms of $x(t)$ we find

$$x(t) = \frac{\ell}{2} + A\cos(\omega_0 t) + B\sin(\omega_0 t). \tag{9}$$

Imposing the initial conditions Eq.(5), yields $B = 0$ and $A = \ell/2$. Therefore, we finally have the solution

$$x(t) = \frac{\ell}{2}\left(1 - \cos(\omega_0 t)\right) = \ell\sin^2(\omega_0 t/2), \tag{10}$$

with ω_0 again given by $\sqrt{4k/m}$ and where we used a trigonometric identity in the final line.

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3. **(10 points) Removing a spring**

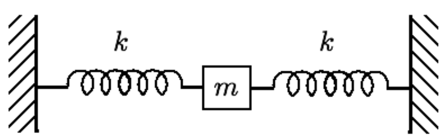

Figure 2

We again want to find the specific solution for $x(t)$ given initial conditions we can infer from the system. Namely, a mass connected to two springs of spring constant k is moving with amplitude d (Fig. 2). Such a system has an effective spring force of $k + k = 2k$ because the mass is in equilibrium in Fig. 2 and thus any deviation from $x = 0$ returns it to its original position. At the time when the particle is at $x = d/2$ and is moving to the right, we cut the spring on the right, and we are to determine the mass's subsequent motion and the amplitude of that motion.

For a mass moving with amplitude d, the position of the particle (prior to removing the spring) is

$$x(t) = d\sin(\omega_1 t - \phi), \tag{11}$$

where $\omega_1 = \sqrt{2k/m}$ and ϕ is an unknown (and ultimately unimportant) phase. If at a $t = t_0$ we find $x(t_0) = d/2$, then we must have $\sin(\omega_1 t_0 - \phi) = 1/2$. We can thus infer that

$$\cos(\omega_1 t_0 - \phi) = \sqrt{1 - \sin^2(\omega_1 t_0 - \phi)} = \sqrt{1 - (1/2)^2} = \frac{\sqrt{3}}{2}, \tag{12}$$

where we chose the positive root to ensure that the velocity is positive at this chosen time. We thus find that the velocity is

$$\dot{x}(t_0) = d\omega_1 \cos(\omega_1 t - \phi) = \frac{d\omega_1\sqrt{3}}{2}. \tag{13}$$

Now, we consider the system after the right spring is removed. In this new system, the angular frequency of motion is $\omega_2 = \sqrt{k/m}$ (because there is only a single spring). We shift our origin of time so that t_0 is now $t = 0$. We thus have the initial conditions

$$x(t=0) = \frac{d}{2}, \qquad \dot{x}(t=0) = \frac{d\omega_1\sqrt{3}}{2}. \tag{14}$$

Given the general solution of the harmonic oscillator (stated in the notes) in terms of the initial position and velocity, we have

$$\begin{aligned} x(t) &= x_0\cos(\omega_2 t) + \frac{v_0}{\omega_2}\sin(\omega_2 t) \\ &= \frac{d}{2}\cos(\omega_2 t) + d\,\frac{\omega_1}{\omega_2}\frac{\sqrt{3}}{2}\sin(\omega_2 t). \end{aligned} \tag{15}$$

Given $\omega_2 = \sqrt{k/m}$ and $\omega_1 = \sqrt{2k/m}$, this result then reduces to

$$x(t) = \frac{d}{2}\cos(\omega_2 t) + d\sqrt{\frac{3}{2}}\sin(\omega_2 t) \tag{16}$$

which is the final result for the value of the position if $t = 0$ is the time when the right spring is removed.

To compute the amplitude, we note that when we have a sinusoidal motion represented by a linear combination of a sine and cosine function, the amplitude of the motion is the square root of the sum of squares of the coefficients of the sinusoids. In this case, the amplitude is

$$\text{Amplitude} = \sqrt{\left(\frac{d}{2}\right)^2 + \left(d\sqrt{\frac{3}{2}}\right)^2} = d\sqrt{\frac{1}{4} + \frac{3}{2}} = d\frac{\sqrt{7}}{2}. \tag{17}$$

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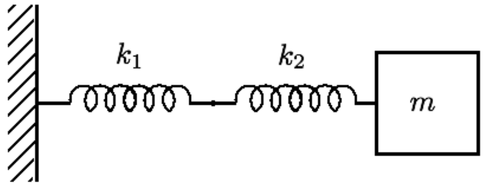
4. **(5 points) Effective Spring Constant**

We are seeking to find the effective spring constant experienced by the mass in Eq.(**??**) presuming it is translated a distance x from its equilibrium configuration. The easiest way to do this is to assume there exists a very small mass Δm at the connection point between k_1 and k_2, determine the equations of motion of the system and then take $\Delta m \to 0$ to find a useful constraint. We will define the position of Δm as x_Δ.



(a)

Figure 3

At equilibrium we have $x = x_\Delta = 0$. When we pull the mass m to the right, both x and x_Δ increase and the net forces on m and Δm yield the equations, respectively,

$$m\ddot{x} = -k_2(x - x_\Delta) \tag{18}$$

$$\Delta m \ddot{x}_\Delta = +k_2(x - x_\Delta) - k_1 x_\Delta. \tag{19}$$

Eq.(18) follows from considering what force m experiences when x is increased beyond x_Δ. The first term in Eq.(19) follows from a similar consideration and the second term follows from the fact that Δm is attached to a spring on its left.

In this system, there is in fact no Δm, so we take $\Delta m \to 0$, thus yielding the equation

$$0 = k_2(x - x_\Delta) - k_1 x_\Delta = k_2 x - (k_2 + k_1)x_\Delta, \tag{20}$$

which when solved for x_Δ gives

$$x_\Delta = \frac{k_2}{k_1 + k_2} x. \tag{21}$$

Inserting this into Eq.(18) yields

$$m\ddot{x} = -k_2\left(x - \frac{k_2}{k_1 + k_2}x\right) = -\frac{k_1 k_2}{k_1 + k_2}x \equiv -k_{\text{eff}} x, \tag{22}$$

implying the effective constant is $k_{\text{eff}} = k_1 k_2/(k_1 + k_2)$ for the system in Fig. 3a.

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5. **(10 points) Oscillation of bead with gravitating masses**

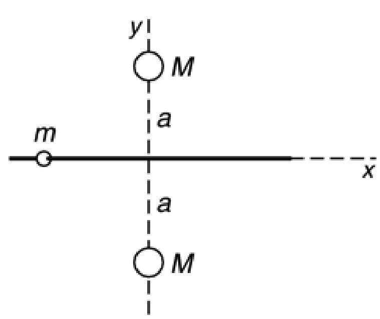

Figure 4

We want to determine the position as a function of time of the mass shown in Fig. **??** given that it under goes small oscillations about the origin with the initial conditions $x = 0$ and $\dot{x}(0) = v_0$. In order to determine $x(t)$ we first need to find the equation of motion of the system under the stated approximation.

The easiest way to determine this equation of motion is to find the potential energy of the system, approximate it given an assumption of small oscillations, and then use the approximation to find the force.

Given that the gravitational potential energy between two masses m_1 and m_2 separated by a distance r is

$$U = -\frac{Gm_1 m_2}{r}, \tag{23}$$

where G is Newton's gravitational constant. For the system in Fig. **??**, if the mass is at a position x along the x axis, then the distance between m and each mass M is $\sqrt{a^2 + x^2}$. Thus the total potential energy of the system is

The potential energy of the mass at a position x is

$$U(x) = -\frac{GMm}{\sqrt{a^2 + x^2}} - \frac{GMm}{\sqrt{a^2 + x^2}} = -\frac{2GMm}{a}\left(1 + \frac{x^2}{a^2}\right)^{-1/2}. \tag{24}$$

Now to implement the small oscillations approximation, we will make the assumption that the oscillating mass m stays very close to the origin. Namely, we will take

$$|x| \ll a \qquad \text{[Small oscillations]}. \tag{25}$$

This will allow us to approximate Eq.(24) given the Taylor series approximation

$$(1 + x)^n = 1 + nx + \mathcal{O}(x^2) \qquad \text{[For } |x| \ll 1 \text{ and } n \text{ any real number]}. \tag{26}$$

We thus find

$$U(x) = -\frac{2GMm}{a}\left(1 + \frac{x^2}{a^2}\right)^{-1/2} = -\frac{2GMm}{a}\left(1 - \frac{x^2}{2a^2} + \mathcal{O}(x^4)\right). \tag{27}$$

From the approximate potential energy Eq.(27), we could then find the force exerted on the mass in the small oscillation approximation. With the standard equation expressing the relationship between force and potential energy, we find

$$F(x) = -\frac{dU}{dx} = -\frac{2GMm}{a^3}x \tag{28}$$

Therefore, the small-oscillations equation of motion for this system is

$$m\ddot{x} = F(x) = -\frac{2GMm}{a^3}x \tag{29}$$

or, equivalently,

$$\ddot{x} + \omega_0^2 x = 0, \tag{30}$$

where we defined

$$\omega_0 \equiv \sqrt{\frac{2GM}{a^3}}, \tag{31}$$

as the frequency of small oscillations of the system. The general solution to Eq.(**??**) given initial conditions $x(0) = x_0$ and $\dot{x}(0) = x_0$ is

$$x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t). \tag{32}$$

With the stated initial conditions (specifically of x_0), we then find Eq.(32) becomes

$$x(t) = \frac{v_0}{\omega_0} \sin(\omega_0 t) = v_0 \sqrt{\frac{a^3}{2GM}} \sin\left[t\sqrt{\frac{2GM}{a^3}}\right] \tag{33}$$

As a side note, given Eq.(33) we thus also see that the condition for small oscillations (i.e., $|x| \ll a$) is only satisfied if the initial speed satisfies

$$v_0 \ll \sqrt{\frac{2GM}{a}}. \tag{34}$$

6. **Numerical Solution to Differential Equations**
Euler's method is a numerical procedure for solving differential equations which makes use of the first order changes in functions. Namely for a time ϵ which is small relative to the larger time scales we are interested in, we can make the approximation

$$x(t+\epsilon) = x(t) + \epsilon\, \dot{x}(t) + \mathcal{O}(\epsilon^2) \tag{35}$$

$$\dot{x}(t+\epsilon) = \dot{x}(t) + \epsilon\, \ddot{x}(t) + \mathcal{O}(\epsilon^2), \tag{36}$$

where $\mathcal{O}(\epsilon^2)$ stands for terms of order ϵ^2 or higher. Therefore, if we have an equation of motion for $x(t)$ which is written as

$$\ddot{x}(t) = F(x, \dot{x}, t), \tag{37}$$

where F is a function of position x, velocity $\dot{x}$, and/or time t, we can solve for $x(t)$ step-by-step. Assuming we know $x(t)$ and $\dot{x}(t)$, at $t = 0$ we can use Eq.(35) to find x at $t = 0 + \epsilon$ and we can use Eq.(36) and Eq.(37) to find $\dot{x}$ at $t = 0 + \epsilon$. In this way, we can find $x(t)$ from $t = 0$ to an arbitrary time t by moving in steps of ϵ. In this problem we implement this procedure to solve the equation of motion of a pendulum

Before we can implement this code we must get set up with our numerical program *Mathematica*. Here are the preliminary steps before you can begin this problem

(i) Log in to your account in MIT's Athena Cluster, and go to the course website.

(ii) Download the code "numerical_diff_eq.nb" from the course webpage and open it in *Mathematica*.

(iii) Select a block of code and run it by pressing `Shift+Enter`.

Now we can begin the problem.

(a) **(10 points)** For each line of the code, write a sentence explaining the line's utility in the overall code. (You can annotate the code itself)

(b) **(5 points)** A student wrote this code intending to produce a plot of simple harmonic motion, but he made a few errors. Given Eq.(35), Eq.(36), and the simple harmonic equation of motion

$$\ddot{x} = -\omega_0^2 x \tag{38}$$

modify the code to plot $x(t)$ as a function of time with the initial conditions $x(t = 0) = 1.0$ m, $\dot{x}(t = 0) = 1.0$ m/s^2, and with $\omega_0^2 = 5.0$ rad/s^2. (*Hint: The simplest way to do this is to just copy and paste the incorrect code and make modifications to make it correct.*)

(c) **Extra Credit: (+5 points)** Taking $\theta(t = 0) = \pi/2$ rad, $\dot{\theta}(t = 0) = 0$, $g = -9.8$ m/s^2, modify the Euler's algorithm code to plot $\theta(t)$ as a function of time for the pendulum equation of motion

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta. \tag{39}$$

Useful Information: The *Mathematica* function for $\sin(x)$ is "`Sin[x]`". What is the period of the pendulum motion in the plot? (i.e., how much time does it take to go from one amplitude to another?). How does this compare to the prediction given by $T = 2\pi\sqrt{\ell/g}$?

Submitting: As your submission for this part of the assignment, you can print out the entire *Mathematica* notebook which should include your annotations of the existing code, and your plots of simple harmonic and pendulum motion (in addition to the answers for the questions in (c)).

7. **(15 points) Small Oscillations about equilibria**
Given the potential energy function

$$U(x) = \frac{E_0}{a^4}\left(x^4 + 4ax^3 - 8a^2x^2\right), \tag{40}$$

there are a number of questions we can ask concerning the oscillatory behavior of this system. Here are a few of them:

Possible Questions

- At what values of x is the system at an equilibrium?
- At which of these values is the equilibrium stable?
- What are the frequencies of small oscillations about these equilibrium points? (Also, how do we define small oscillations in this system?)
- What are corrections to the equation of motion about the equilibria?
- If the mass begins at an unstable equilibrium, about how much time would it take it to fall into one of the stable equilibria?
- What is the total energy of this system near the various stable equilibria?

We will answer the first three questions. First, we note that a particle is at an equilibrium (but not necessarily a stable equilibrium) if the net-force acting on the particle is zero. Given the fact that the force is the negative of the spatial derivative of the potential energy, we find that the force on the particle is

$$F(x) = -U'(x) = -\frac{E_0}{a^4}\left(4x^3 + 12ax^2 - 16a^2x\right) = -\frac{4E_0}{a^4}x\left(x^2 + 3ax - 4a^2\right). \tag{41}$$

Setting this force to zero, we have the condition

$$x\left(x^2 + 3ax - 4a^2\right) = x\left(x + 4a\right)\left(x - a\right) = 0, \tag{42}$$

which indicates that the equilibria for the potential energy Eq.(40) are

$$x = -4a,\ 0,\ a \qquad \text{[Equilibria of } U(x)\text{]}. \tag{43}$$

Now to determine whether these equilibria are stable equilibria, we need to determine whether they correspond to local minima of the potential energy. To do so, we apply the second derivative test to these equilibria which amounts to evaluating the second derivative of the potential energy at each value; results which are greater than zero are local minima and stable equilibria.

Computing the second derivative of Eq.(40), we find

$$U''(x) = \frac{E_0}{a^4}\left(12x^2 + 24ax - 16a^2\right) = \frac{2E_0}{a^4}\left(6x^2 + 12ax - 8a^2\right) \tag{44}$$

and evaluating this result at Eq.(43), gives us

$$U''(x=-4a) = \frac{80E_0}{a^2} > 0 \qquad U''(x=0) = -\frac{16E_0}{a^2} < 0, \qquad U''(x=a) = \frac{20E_0}{a^2} > 0. \tag{45}$$

Thus we see that $x = -4a$ and $x = a$ are stable equilibria but $x = 0$ is an unstable equilibrium.

Finally, we can compute the frequency of oscillations about each stable equilibrium x_{eq} by using the general result

$$\omega_0 = \sqrt{\frac{U''(x_{\text{eq}})}{m}}, \tag{46}$$

where we assume the oscillating degree of freedom is a mass m which is moving linearly. Given Eq.(45), the oscillation frequencies are then

$$\omega_0 = \sqrt{\frac{80E_0}{ma^2}}, \qquad [\text{About } x = -4a] \quad \text{and} \quad \omega_0 = \sqrt{\frac{20E_0}{ma^2}}, \qquad [\text{About } x = a]. \tag{47}$$

Around each of the stable equilibria the equation of motion of the mass m is (approximately)

$$\ddot{x} + \omega_0^2(x - x_{\text{eq}}) = 0, \tag{48}$$

where we substitute in the respective ω_0 and x_{eq} for each point.

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