Solutions 3: Damped and Forced Oscillators (Midterm Week)

Due Wednesday June 28, at 9AM under Rene García's door

Preface: This problem set provides practice in understanding damped harmonic oscillator systems, solving forced oscillator equations, and exploring numerical solutions to differential equations.

1. **Equations of Underdamping**

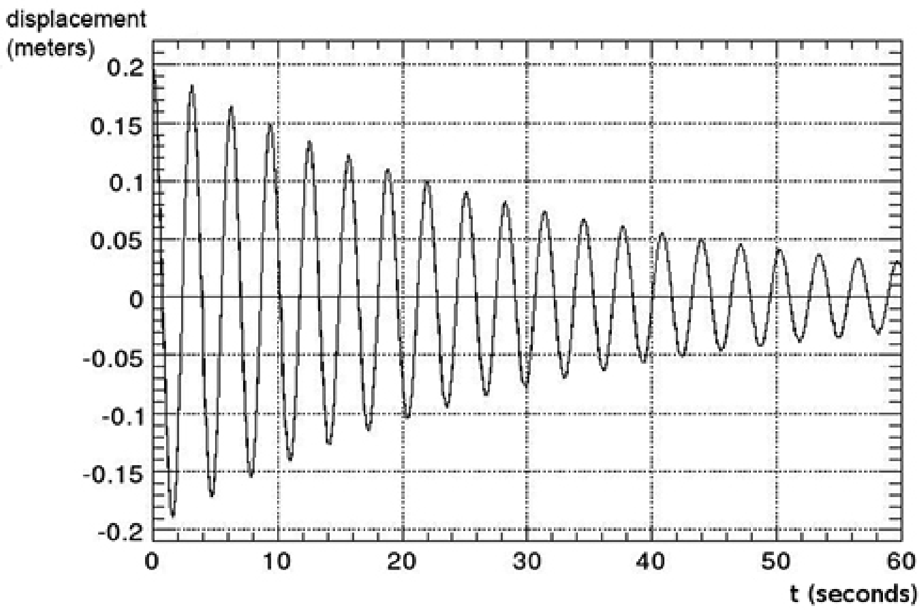

Figure 1: Damped Oscillator

(a) Given the information in the prompt and in the plot we want to determine the mass of the oscillating degree of freedom and the quality factor of the damped oscillation. The prompt tells us we apply a force $F_{\text{app}} = 4$ N to pull a spring (of unknown spring constant k) a distance $\Delta x = 0.2$ m. By Hooke's law, we find that the spring constant of the system

$$k = \frac{F_{\text{app}}}{\Delta x} = 20\ \text{N/m}. \tag{1}$$

The plot provides us with two pieces of information not included in the prompt. From the prompt, we can determine the period of oscillation and we can determine how much time it takes the amplitude to decay to a specific fraction of its initial value. From the figure we can estimate the period to be $T = 3$ s. For an underdamped oscillator, the period of motion is

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}, \tag{2}$$

where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$ with $\omega_0^2 = k/m$. Noting that it takes $t_{1/2} = 22$ s, for the amplitude of oscillation to decay to half of its initial value we have

$$Ae^{-\gamma t_{1/2}} = A(0.5), \tag{3}$$

which upon taking the logarithm of both sides and dividing by $t_{1/2}$ yields

$$\gamma = \frac{\ln 2}{t_{1/2}} \simeq 0.0315 \text{ s}^{-1}. \tag{4}$$

With a knowledge of k, T, and γ, as provided by Eq.(1), the plot, and Eq.(4), respectively, we can determine the mass through Eq.(2). Solving for ω_0^2 given Ω^2, we find

$$\omega_0^2 = \frac{k}{m} = \Omega^2 + \gamma^2. \tag{5}$$

Then using Eq.(2) to write Ω in terms of the period and Eq.(4) to write γ in terms of the "half-life" of the oscillation, we find

$$m = \frac{k}{4\pi^2/T^2 + (\ln 2/t_{1/2})^2} \approx 4.56 \text{ kg}. \tag{6}$$

(b) With Eq.(6) and Eq.(1), we find that the natural angular frequency of motion is

$$\omega_0 = \sqrt{\frac{k}{m}} \approx 2.09 \text{ rad/s}. \tag{7}$$

Therefore, with Eq.(**??**) and Eq.(4), we find that the quality factor of the system is

$$Q = \frac{\omega_0}{2\gamma} \approx 33. \tag{8}$$

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2. **R factor**

For an underdamped oscillator, Amy defines the "R factor" as

$$R = \pi \times (\text{Number of oscillation cycles it takes to reach } 1/e \text{ of the initial amplitude}). \tag{9}$$

How does R compare to the quality factor Q for an underdamped oscillator? (We're considering a very weakly damped oscillator)

Solution: We want to compute R and thereby determine how R relates to Q for an underdamped oscillator. By the end of the problem, we will have found a new way to interpret Q in terms of kinematic variables rather than energy.

For an underdamped oscillator, the position as a function of time is

$$x(t) = Ae^{-\gamma t}\cos(\Omega t - \phi), \tag{10}$$

where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$. By definition, the Q factor for this oscillator is $Q = \omega_0/2\gamma$. In order to find the number of oscillation cycles it takes to reach $1/e$ of the initial amplitude, we first need to determine how much time it takes to reach e^{-1} of the initial amplitude. By the fact that the amplitude has the time dependent $A(t) = A_0 e^{-\gamma t}$, we can find the time it takes to satisfy the condition in the parentheses of Eq.(9) with

$$Ae^{-\gamma t_1} = Ae^{-1}. \tag{11}$$

Solving for t_1, we then find that $t_1 = 1/\gamma$. Now, given that a single oscillation cycle occurs over a time $T = 2\pi/\Omega \simeq 2\pi/\omega_0$ (where we took the approximation given the assumption of very weakly damped motion), the number of period cycles we experience in time t_1 is

$$\#\text{ of oscillation cycles} = \frac{t_1}{T} \simeq \frac{\omega_0}{2\pi\gamma}. \tag{12}$$

Therefore, Eq.(9) becomes

$$R = \pi \times \#\text{ of oscillation cycles} \simeq \frac{\omega_0}{2\gamma}, \tag{13}$$

for very weakly damped motion. Given $Q = \omega_0/2\gamma$, we therefore find $R \simeq Q$.

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3. **Forced Oscillator**

(a) Our goal in this problem is to determine $x(t)$ given the above initial conditions. First, we write the equation of motion for the system. Using the identity $\sin^2 x = 1 - \cos(2x)$, we find the external force can be written as

$$F(t) = F_0 \sin^2(\omega t) = \frac{F_0}{2}(1 - \cos(2\omega t)). \tag{14}$$

Thus, Newton's 2nd law for the system gives us

$$m\ddot{x} = -kx + \frac{F_0}{2}(1 - \cos(2\omega t)), \tag{15}$$

which can be written as

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{2m} - \frac{F_0}{2m}\cos(2\omega t), \tag{16}$$

where $\omega_0^2 = k/m$. Defining

$$X \equiv x - \frac{F_0}{2m\omega_0^2}, \tag{17}$$

we can write Eq.(16) as

$$\ddot{X} + \omega_0^2 X = -\frac{F_0}{2m}\cos(2\omega t). \tag{18}$$

From here we can use the general solution to the driven undamped equation of motion to write the general solution for this case. Considering Eq. (4) and Eq. (11) in Lecture notes 04, we find that the solution to Eq.(18) is

$$X(t) = B\cos(\omega_0 t) + C\sin(\omega_0 t) - \frac{F_0/2m}{\omega_0^2 - 4\omega^2}\cos(2\omega t). \tag{19}$$

Given Eq.(17), we can write this solution in terms of $x(t)$ as

$$x(t) = \frac{F_0}{2m\omega_0^2} + B\cos(\omega_0 t) + C\sin(\omega_0 t) - \frac{F_0/2m}{\omega_0^2 - 4\omega^2}\cos(2\omega t). \tag{20}$$

Imposing the initial condition $x(t = 0) = 0$ and $\dot{x}(t = 0) = 0$, we find

$$B = \frac{F_0}{2m}\left[\frac{1}{\omega_0^2 - 4\omega^2} - \frac{1}{\omega_0^2}\right], \qquad C = 0. \tag{21}$$

Therefore, the general solution to the given differential equation is

$$x(t) = \frac{F_0/2m}{\omega_0^2 - 4\omega^2}\left(\cos(\omega_0 t) - \cos(2\omega t)\right) + \frac{F_0}{2m\omega_0^2}(1 - \cos(\omega_0 t)). \tag{22}$$

or

$$x(t) = \frac{F_0/2m}{\omega_0^2 - 4\omega^2}\left(\cos(\omega_0 t) - \cos(2\omega t)\right) + \frac{F_0}{m\omega_0^2}\sin^2(\omega_0 t) \tag{23}$$

(b) Resonance is defined as the state in which the amplitude of oscillation is maximized. Resonance is defined by the nonzero frequency which yields this maximum.The only term in Eq.(22) whose amplitude is maximized by a non-zero value of ω is the first term. To maximize the coefficient of the first term, we minimized the denominator, and we can do so by setting $\omega_0^2 = 4\omega^2$. Given $k = m\omega_0^2$, we find that the value of k which puts the system in resonance is

$$k = 4\omega_0^2 \tag{24}$$

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4. **Numerical Solution to Differential Equations - Part II**

(a) The various relationships between ω_0 and γ which determine the type of motion of our damped oscillatory are as follows:

$$\omega_0 > \gamma \quad \text{leads to underdamped motion} \tag{25}$$

$$\omega_0 < \gamma \quad \text{leads to damped motion} \tag{26}$$

$$\omega_0 = \gamma \quad \text{leads to critically damped motion} \tag{27}$$

(b) Mathematica plots on website

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5. **Ball in Bowl**

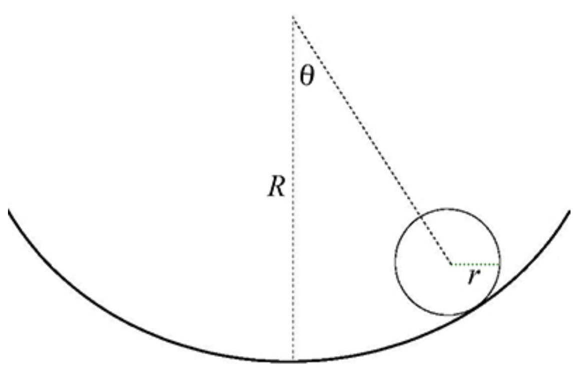

Figure 2: Ball in Bowl

State (but do not answer) **three** precise physics questions we can ask about this system.

Solution:

There are many different questions we can ask about this system. Some of the questions below can be answered using methods, we have already covered in class. Other questions require methods outside the class (which is fine because I'm only asking you to *ask* questions and not answer them).

– What is the equation of motion of the ball in the bowl in terms of $\theta(t)$

– What is the frequency of small oscillations about the point $\theta = 0$?

– As an integral, what is an expression for the total period of motion presuming the ball begins from rest at an angle θ_0

– What is the first non-linear approximation to the simple harmonic oscillator equation of motion for this system?

– At what radius of the smaller sphere does the angular frequency of motion disappear?

– If we considered the bowl and sphere to be three-dimensional what are the resulting equations of motion?

– If we were to shake the bowl vertically at frequency ω, such that the gravitational acceleration became $g(t) = g + g_0 \cos(\omega t)$? for some quantity g_0, what would the resulting equation of motion be, and what would be the solution to that equation of motion?