Solutions 4: Coupled Oscillations and Fourier Series

(Formally) Due Wednesday July 5, at 9AM under Rene García's door

Preface: In this assignment, we practice solving differential equations, studying the nonlinear properties of the pendulum. practice computing the properties of coupled oscillator systems, and use Fourier Series to derive an identity for π^2.

1. **(10 points) Third-order differential equation**

   We want to find the three possible solutions (and ultimately the general solution) to the differential equation

$$\frac{d^3}{dt^3}x(t) + x(t) = 0. \tag{1}$$

   We begin by guessing the solution

$$x(t) = Ae^{\alpha t}.$$

   Inserting this solution into Eq.(1), we obtain the condition

$$\alpha^3 + 1 = 0. \tag{2}$$

   By an algebraic identity this equation can be reduced to

$$\alpha^3 + 1 = (\alpha + 1)(\alpha^2 - \alpha + 1) = 0. \tag{3}$$

   From the first expression in the parentheses, we find one solution is $\alpha_0 = -1$. The other two solutions can be found by using the quadratic formula on the quantity in the parentheses. Doing so, we find that the three solutions for α are

$$\alpha_0 = -1, \quad \alpha_+ = \frac{1 - i\sqrt{3}}{2}, \quad \alpha_+ = \frac{1 + i\sqrt{3}}{2}. \tag{4}$$

   Therefore, given our original guess $x(t) = Ae^{\alpha t}$, we find that the three independent solutions are

$$\boxed{e^{-t}, \quad e^{(1-i\sqrt{3})t/2}, \quad e^{(1+i\sqrt{3})t/2}.} \tag{5}$$

   Writing the general solution as a linear combination of these results we have

$$x(t) = Ae^{-t} + Be^{(1-i\sqrt{3})t/2} + Ce^{(1+i\sqrt{3})t/2}, \qquad \text{[Complex Solution]} \tag{6}$$

   for arbitrary complex constants A, B, and C. We can find the real solutions to Eq.(1) by taking the real part of Eq.(6). First defining A, B and C in terms of their real and imaginary parts we have

$$A = A_1 + iA_2, \quad B = B_1 + iB_2, \quad C = C_1 + iC_2. \tag{7}$$

   Using the identity

$$\text{Re}\left[(a_1 + ia_2)(b_1 + ib_2)\right] = a_1 b_1 - a_2 b_2, \tag{8}$$

   we then find that the real part of Eq.(6) given the definitions in Eq.(7) is

$$\boxed{x(t) = A_1 e^{-t} + e^{t/2}\left(B_0 \cos(t\sqrt{3}/2) + C_0 \sin(t\sqrt{3}/2)\right), \qquad \text{[Real Solution]}} \tag{9}$$

where A_1 is real and we defined the real constants $B_0 \equiv B_1 + C_1$ and $C_0 \equiv B_2 - C_2$.

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2. **Corrections to Pendulum Period**

(a) **(10 points)** We want to find an integral expression for the exact period of a pendulum which begins at rest from an arbitrary angle. If the pendulum has an amplitude of θ_0 then the energy of the system is

$$E_0 = mg\ell(1 - \cos\theta_0). \tag{10}$$

In general, the energy of the pendulum swinging with angular velocity $\dot{\theta}$ and at an angle θ is

$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1 - \cos\theta) \tag{11}$$

By conservation of energy, we have $E_0 = E$, or

$$mg\ell(1 - \cos\theta_0) = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell(1 - \cos\theta). \tag{12}$$

Solving this equation for $\dot{\theta}$, we obtain

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}}\sqrt{\cos\theta - \cos\theta_0}, \tag{13}$$

where the sign of the root is taken to be positive, in order to be consistent with out later integration from 0 to θ_0. The period of the pendulum is $4\times$ the amount of time it takes the bob to go from 0 to θ_0. So, with the fact that $\dot{\theta} = d\theta/dt$ and that

$$\frac{dt}{d\theta} = \frac{1}{\dot{\theta}}, \tag{14}$$

we find the period of the pendulum is

$$\int_0^{T/4} dt = \int_0^{\theta_0} \frac{dt}{d\theta}\, d\theta = \int_0^{\theta_0} \frac{d\theta}{\dot{\theta}}. \tag{15}$$

Given Eq.(13), we thus obtain

$$\begin{aligned} T &= 4\int_0^{\theta_0} \frac{d\theta}{\dot{\theta}} \\ &= 4\sqrt{\frac{\ell}{2g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \\ &= \sqrt{\frac{8\ell}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \end{aligned} \tag{16}$$

(b) **(10 points)** In this part, we want to express Eq.(16) in such a way that the $|\theta_0| \ll 0$ approximation clearly leads to the small-angle result $T \simeq 2\pi\sqrt{\ell/g}$. Using the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$, Eq.(16) becomes

$$T = 2\sqrt{\frac{2\ell}{g}}\int_0^{\theta_0} \frac{1}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}\, d\theta. \tag{17}$$

Changing variables to $\sin x = \sin(\theta/2)/\sin(\theta_0/2)$, we find that $x = \pi/2$ when $\theta = \theta_0/2$ and $x = 0$ when $\theta = 0$. We also find that the differential element in Eq.(17) transforms as

$$\begin{aligned} d\theta &= \frac{2\sin(\theta_0/2)\cos x}{\cos(\theta/2)}\,dx \\ &= \frac{2\sqrt{\sin^2(\theta_0/2)-\sin^2(\theta/2)}}{\sqrt{1-\sin^2(\theta/2)}}\,dx, \end{aligned} \tag{18}$$

where, to obtain the second line, we used the identity

$$\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{\sin^2(\theta/2)}{\sin^2(\theta_0/2)}}. \tag{19}$$

Thus, Eq.(17) becomes

$$\begin{aligned} T &= 2\sqrt{\frac{2\ell}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{\sin^2(\theta_0/2)-\sin^2(\theta/2)}}\frac{2\sqrt{\sin^2(\theta_0/2)-\sin^2(\theta/2)}}{\sqrt{1-\sin^2(\theta/2)}}\,dx \\ &= 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\frac{1}{\sqrt{1-\sin^2(\theta_0/2)\sin^2 x}}\,dx. \end{aligned} \tag{20}$$

Now, given the Taylor series

$$(1+x)^n = 1 + nx + \mathcal{O}(x^2), \tag{21}$$

we can approximate Eq.(20), for $\sin^2\theta_0$ sufficiently small, as

$$\begin{aligned} T &= 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}\sin^2\theta_0\sin^2 x+\cdots\right)dx \\ &= 4\sqrt{\frac{\ell}{g}}\left(\frac{\pi}{2}+\frac{\theta_0^2}{2}\frac{\pi}{4}+\cdots\right). \end{aligned} \tag{22}$$

Where we used the approximation $\sin\theta_0 \simeq \theta_0$ and the integral

$$\int_0^{\pi/2}\sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2}(1-\cos(2x))\,dx = \frac{\pi}{2} \tag{23}$$

in the final line. Factoring out a $\pi/2$ from inside the parentheses in the final line of Eq.(22), we find which becomes

$$T = 2\pi\sqrt{\frac{\ell}{g}}\left(1+\frac{\theta_0^2}{16}+\cdots\right), \tag{24}$$

the desired result.

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3. **(15 points) Two coupled oscillators**

We want to find the most general solutions to the equations of motion for x_1 and x_2. We will find these general solutions for the given coordinate variables, by first finding the general solutions for the

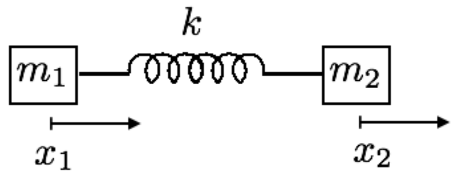

Figure 1

equations of motion of X_{cm} and R.

First, writing down the equations of motion for this system, we have

$$m_1\ddot{x}_1 = -k(x_2 - x_1) \tag{25}$$
$$m_2\ddot{x}_2 = +k(x_2 - x_1). \tag{26}$$

Adding these two equations, gives us

$$m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0, \tag{27}$$

which, by the definition of X_{cm} as $X_{\text{cm}} = (m_1x_1 + m_2x_2)/(m_1 + m_2)$, implies

$$\ddot{X}_{\text{cm}} = 0. \tag{28}$$

Since X_{cm} has zero acceleration, it represents a position which moves uniformly in space. Therefore, we know (either by integrating Eq.(28) twice or by our understanding of kinematics)

$$X_{\text{cm}}(t) = X_{\text{cm}}(0) + V_{\text{cm}}t. \tag{29}$$

Now, we will find the general solution to the equation of motion of R. Given the system of equations

$$X_{\text{cm}} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \tag{30}$$
$$R = x_2 - x_1, \tag{31}$$

we can solve for x_1 and x_2. Multiplying the first equation by $(m_1+m_2)/m_1$ and adding it to the second equation, we find

$$\frac{m_1 + m_2}{m_1}X_{\text{cm}} + R = \left(1 + \frac{m_2}{m_1}\right)x_2, \tag{32}$$

or

$$x_2 = X_{\text{cm}} + \frac{\mu}{m_2}R. \tag{33}$$

and with $x_1 = x_2 - R$ we find

$$x_1 = X_{\text{cm}} - \frac{\mu}{m_1}R \tag{34}$$

Inserting these representations of x_1 and x_2 into Eq.(25) and Eq.(26) (and using $\ddot{X}_{\text{cm}} = 0$), we find, respectively,

$$-\mu\ddot{R} = k\mu\left(\frac{1}{m_2} + \frac{1}{m_1}\right)R \tag{35}$$
$$+\mu\ddot{R} = -k\mu\left(\frac{1}{m_2} + \frac{1}{m_1}\right)R, \tag{36}$$

which are the same equation of motion. We thus have

$$\ddot{R} + \frac{k}{\mu}R = 0, \tag{37}$$

which has the general solution

$$R(t) = R_{\text{amp.}} \cos(\omega_0 t - \phi), \tag{38}$$

where

$$\omega_0 = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}. \tag{39}$$

Thus, with Eq.(29) and Eq.(38), Eq.(34) and Eq.(33) become

$$x_1(t) = X_{\text{cm}}(0) + V_{\text{cm}}t - \frac{m_2}{m_1 + m_2}R_{\text{amp.}} \cos(\omega_0 t - \phi) \tag{40}$$

$$x_2(t) = X_{\text{cm}}(0) + V_{\text{cm}}t + \frac{m_1}{m_1 + m_2}R_{\text{amp.}} \cos(\omega_0 t - \phi), \tag{41}$$

which are the general solutions to the original system of equations of motion. These equations state that the coupled mass system has a center of mass which moves uniformly at a speed V_{cm} while the two masses oscillate in an accordion-like fashion. Considering limit cases, for $m_1 \gg m_2$, x_1 reduces to uniform motion with no oscillation. This makes sense since a very heavy mass would not be too affected by the dynamics of a much lighter mass which is coupled to it by a spring. Also, in this limit the frequency of oscillation of the lighter mass reduces to $\sqrt{k/m_2}$ because the larger mass acts effectively like a fixed wall.

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4. **Three coupled masses**

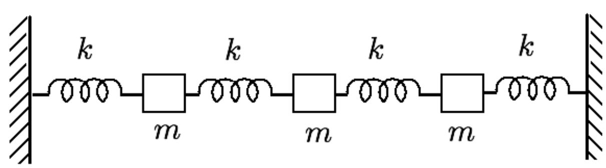

Figure 2

(a) **(15 points)** Our objective is to find the normal modes and normal frequencies of the motion. Writing down the equation of motion as a matrix equation, we have

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{42}$$

where $\omega_0 = \sqrt{k/m}$. With the guess

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{\alpha t}, \tag{43}$$

we have the eigenvalue-eigenvector equation

$$\begin{pmatrix} -2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \tag{44}$$

or

$$\begin{pmatrix} -2\omega_0^2 - \alpha^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 - \alpha^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 - \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0. \tag{45}$$

To find the value of α^2, which leads to non-trivial normal mode solutions we set the determinant of the 3×3 matrix to zero. Doing so, we find

$$\begin{aligned} 0 &= \begin{vmatrix} -2\omega_0^2 - \alpha^2 & \omega_0^2 & 0 \\ \omega_0^2 & -2\omega_0^2 - \alpha^2 & \omega_0^2 \\ 0 & \omega_0^2 & -2\omega_0^2 - \alpha^2 \end{vmatrix} \\ &= -(2\omega_0^2 + \alpha^2)\left[(2\omega_0^2 + \alpha^2)^2 - \omega_0^4\right] - \omega_0^2(-(2\omega_0^2 + \alpha^2)\omega_0^2) \\ &= -(2\omega_0^2 + \alpha^2)\left[(2\omega_0^2 + \alpha^2)^2 - 2\omega_0^4\right]. \end{aligned} \tag{46}$$

Thus, the possible values of α^2 (denoted α_1^2, α_2^2, and α_3^2) are

$$\begin{aligned} \alpha_1^2 &= -2\omega_0^2 \\ \alpha_2^2 &= -(2 - \sqrt{2})\omega_0^2 \\ \alpha_3^2 &= -(2 + \sqrt{2})\omega_0^2. \end{aligned}$$

Given the original guess Eq.(43), each of these values of α^2 is associated with a specific frequency (termed the normal mode frequency) of the system. The possible normal mode frequencies are

$$\begin{aligned} \omega_1 &= \omega_0\sqrt{2} \\ \omega_2 &= \omega_0\sqrt{2 - \sqrt{2}} \\ \omega_3 &= \omega_0\sqrt{2 + \sqrt{2}}. \end{aligned}$$

Now we need to find the coefficients A, B, and C associated with the above values of α^2. We find these coefficients by solving Eq.(45) for each α^2.
For $\alpha_1^2 = -2\omega_0^2$, Eq.(45) becomes

$$\begin{pmatrix} 0 & \omega_0^2 & 0 \\ \omega_0^2 & 0 & \omega_0^2 \\ 0 & \omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0., \tag{47}$$

which implies $A = -C$ and $B = 0$. Thus we have the normal mode

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \text{for frequency } \omega_1 = \omega_0\sqrt{2}. \tag{48}$$

For $\alpha_2^2 = -(2-\sqrt{2})\omega_0^2$, Eq.(45) becomes

$$\begin{pmatrix} -\sqrt{2}\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & -\sqrt{2}\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & -\sqrt{2}\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0., \tag{49}$$

which implies $A = B/\sqrt{2} = C$. Thus we have the normal mode

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad \text{for frequency } \omega_2 = \omega_0\sqrt{2-\sqrt{2}}. \tag{50}$$

For $\alpha_2^2 = -(2+\sqrt{2})\omega_0^2$, Eq.(45) becomes

$$\begin{pmatrix} \sqrt{2}\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & \sqrt{2}\omega_0^2 & \omega_0^2 \\ 0 & \omega_0^2 & \sqrt{2}\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0., \tag{51}$$

which implies $A = -B/\sqrt{2} = C$. Thus we have the normal mode

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \qquad \text{for frequency } \omega_2 = \omega_0\sqrt{2+\sqrt{2}}. \tag{52}$$

In summary, we have the following normal mode-frequency pairs

$$\text{normal mode: } \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \text{normal mode frequency: } \omega_1 = \omega_0\sqrt{2} \tag{53}$$

$$\text{normal mode: } \mathbf{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad \text{normal mode frequency: } \omega_2 = \omega_0\sqrt{2-\sqrt{2}} \tag{54}$$

$$\text{normal mode: } \mathbf{v}_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \qquad \text{normal mode frequency: } \omega_3 = \omega_0\sqrt{2+\sqrt{2}} \tag{55}$$

(b) **(5 points)** Given the above listed normal mode-frequency pairs, we find that the general solution for the equation of motion of this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cos(\sqrt{2}\omega_0 t - \phi_1) + A_2 \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \cos((2-\sqrt{2})\omega_0 t - \phi_2)$$
$$+ A_3 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \cos((2+\sqrt{2})\omega_0 t - \phi_3). \tag{56}$$

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5. **(15 points) The Strogatz Sync**

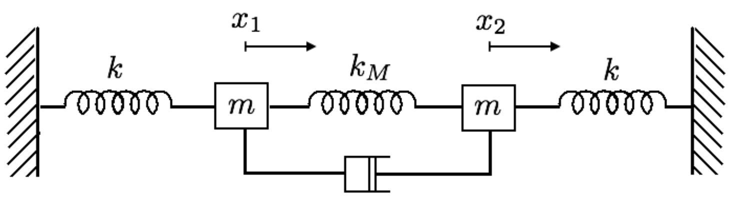

Figure 3

Our goal is to develop a physical model of Strogatz's system using the scenario depicted in Fig. 3. Through the model we hope to explain why the symmetric mode (i.e., the $x_1 = x_2$ mode) remains, while all anti-symmetric motion decays away.

The equations of motion for the system in the figure are

$$m\ddot{x}_1 = -kx_1 + k_M(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) \tag{57}$$
$$m\ddot{x}_2 = -kx_2 - k_M(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1). \tag{58}$$

Dividing by m and adding and subtracting these equations gives us, respectively,

$$\ddot{x}_+ = -\omega_0^2 x_+ \tag{59}$$
$$\ddot{x}_- = -\omega_1^2 x_- - 4\gamma\dot{x}_- \tag{60}$$

where we defined $x_+ \equiv x_2 + x_1$ and $x_- \equiv x_2 - x_1$ and

$$\gamma = \frac{b}{2m}, \qquad \omega_0^2 = \frac{k}{m}, \qquad \omega_1^2 = \frac{k + 2k_M}{m}. \tag{61}$$

Both Eq.(59) and Eq.(60) are equations of motion we have encountered before. They are the simple harmonic oscillator and the damped oscillator equation of motion respectively. Given that our system is oscillating we will assume the damped equation of motion has parameters in the underdamped regime.

The general solution to Eq.(59) is

$$x_+(t) = A_+ \cos(\omega_0 t - \phi_+) \tag{62}$$

and the general solution to Eq.(**??**), for underdamped motion, is

$$x_-(t) = A_- e^{-2\gamma t} \cos(\Omega t - \phi_-), \tag{63}$$

where

$$\Omega \equiv \sqrt{3\omega_1^2 - 2\gamma^2}, \tag{64}$$

and $A_\pm$ and $\phi_\pm$ are arbitrary constants. Given our definitions of x_+ and x_-, we have

$$x_2 = \frac{x_+ + x_-}{2}, \qquad x_1 = \frac{x_+ - x_-}{2}. \tag{65}$$

Therefore the general solutions for x_1 and x_2 (i.e., the solutions with arbitrary initial position and velocity) are

$$x_1 = A_+ \cos(\omega_0 t - \phi_+) - A_- e^{-2\gamma t} \cos(\Omega t - \phi_-) \tag{66}$$

$$x_2 = A_+ \cos(\omega_0 t - \phi_+) + A_- e^{-2\gamma t} \cos(\Omega t - \phi_-), \tag{67}$$

where we reabsorbed the constants $1/2$ into redefinitions of A_+ and A_-. Writing this solution in matrix notation we have

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_0 t - \phi_+) + A_- e^{-2\gamma t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cos(\Omega t - \phi_-). \tag{68}$$

We note that having arbitrary positions and velocities simply means our arbitrary constants in Eq.(68) are not specified. We thus see the general motion is a linear combination of two types of motion: one motion where $x_1 = x_2$ and the masses move in the same direction with the same amplitude (i.e., **symmetric** or "in-sync" motion); and one motion where $x_1 = -x_2$ and the masses move in opposite directions with the same amplitude (i.e., **antisymmetric** motion). The second term in Eq.(68) represents the antisymmetric motion, and thus from this general solution we see that over time the antisymmetric motion decays to zero and all we have left is the symmetric, in-sync, motion.

Physically, the reason this occurs is that the dashpot exerts a retarding force on the masses whenever they are oscillating in opposite directions. There is no such force when the masses are oscillating with the same direction and same amplitude, so over time the motion which pushes against the retarding force is damped away. In the real system shown in the video, the it is the platform upon which the metronomes stand that exerts the retarding force. Similar to the dashpot in the model, the platform exerts a frictional force on the metronomes whenever the metronomes are oscillating in opposite directions. It is possible to study this model more precisely using numerical methods (See [**?**]), but for small oscillation amplitudes the results are essentially those given in Eq.(68).

Now to estimate how much time it would take the system to be in sync, we can estimate how much time it takes the antisymmetric motion to damp away. As an order of magnitude estimate, we can say the antisymmetric mode is no longer relevant when its amplitude is about $1/10$ of its original value (the exact fraction doesn't matter here as long as it is indeed a fraction and not a whole number). Computing the time it takes to reach this value, we have

$$t_1 \approx \frac{\ln 10}{2\gamma} = \frac{m}{b} \ln 10. \tag{69}$$

From this time estimate and the video, we can estimate the value of γ, and thus determine the effective damping and oscillation

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6. **Fourier Series**

(a) **(10 points)** We want to determine the values of α_n and β_n, given the initial conditions stated in the prompt. Because the string begins at rest, we know that its initial velocity $\dot{y}(x, 0)$ is zero. Therefore, by Eq. (48) of Lecture notes 07,

$$\beta_n = \frac{2}{L\omega_m} \int_0^L dx\, \dot{y}(x, 0) \sin\left(\frac{n\pi x}{L}\right), \tag{70}$$

we have $\beta_n = 0$. By Eq. (47) of Lecture notes 07, we have

$$\alpha_n = \frac{2}{L} \int_0^L dx\, y(x, 0) \sin\left(\frac{n\pi x}{L}\right). \tag{71}$$

Given the equation for $y(x,0)$, we then obtain

$$\alpha_n = \frac{2}{L}\int_0^{L/2} dx\, x \sin\left(\frac{n\pi x}{L}\right) + \frac{2}{L}\int_{L/2}^{L} dx\,(L-x)\sin\left(\frac{n\pi x}{L}\right). \tag{72}$$

making the change of variables $x = L - u$ in the second term gives us

$$\begin{aligned}
\alpha_n &= \frac{2}{L}\int_0^{L/2} dx\, x \sin\left(\frac{n\pi x}{L}\right) - \frac{2}{L}\int_{L/2}^{0} du\, u \sin\left(\frac{n\pi}{L}(L-u)\right) \\
&= \frac{2}{L}\int_0^{L/2} dx\, x \sin\left(\frac{n\pi x}{L}\right) + \frac{2}{L}\int_0^{L/2} du\, u \sin\left(n\pi - \frac{n\pi u}{L}\right) \\
&= \frac{2}{L}\int_0^{L/2} dx\, x \sin\left(\frac{n\pi x}{L}\right) - \cos(n\pi)\frac{2}{L}\int_0^{L/2} du\, u \sin\left(\frac{n\pi u}{L}\right) \\
&= \frac{2}{L}\left(1-\cos(n\pi)\right)\int_0^{L/2} dx\, x \sin\left(\frac{n\pi x}{L}\right).
\end{aligned} \tag{73}$$

Now, using the identity

$$\int_{x_1}^{x_2} dx\, x \sin(\alpha x) = \left[-\frac{x}{\alpha}\cos(\alpha x) + \frac{1}{\alpha^2}\sin(\alpha x)\right]_{x_1}^{x_2}, \tag{74}$$

we find

$$\alpha_n = \frac{2}{L}\left(1-\cos(n\pi)\right)\left[-\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\right]. \tag{75}$$

The first term in the brackets is only non-zero for n even. But the coefficient $(1-\cos(n\pi))$ is only non-zero for n odd. Thus, we can ignore the first term in the brackets. For n odd we have $1-\cos(n\pi) = 1-(-1) = 2$, thus we find

$$\alpha_n = \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right), \tag{76}$$

and $y(x,t)$ given the initial conditions is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\omega_n t\right)\sin\left(\frac{n\pi x}{L}\right),} \tag{77}$$

where $\omega_n = n\pi v/L$.

(b) **Extra credit (10 points):** By Eq. (57) in the lecture notes, we have

$$E_{\text{tot}} = \frac{L}{2}\sum_{n=1}^{\infty}\frac{\mu}{2}(\alpha_n^2 + \beta_n^2)\omega_n^2. \tag{78}$$

By the definition of the energy of a string, we have the more general formula

$$E_{\text{tot}} = \frac{1}{2}\int_0^L dx\left[T\left(\frac{\partial y}{\partial x}\right)^2 + \mu\left(\frac{\partial y}{\partial t}\right)^2\right]. \tag{79}$$

At $t = 0$, the string is at rest and $y(x, t)$ is

$$y(x,0) = \begin{cases} x & \text{for } 0 \le x \le L/2 \\ L - x & \text{for } L/2 \le x \le L \end{cases}. \tag{80}$$

Thus, we have

$$\frac{\partial y(x,0)}{\partial x} = \begin{cases} 1 & \text{for } 0 \le x \le L/2 \\ -1 & \text{for } L/2 \le x \le L \end{cases}. \tag{81}$$

So by Eq.(79), the energy of the string at $t = 0$ is

$$E_{\text{tot}} = \frac{1}{2}\int_0^L dx\, T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{T}{2}\left[\int_0^{L/2} dx\,(1)^2 + \int_{L/2}^{L} dx\,(-1)^2\right] = \frac{TL}{2} = \frac{\mu v^2 L}{2}, \tag{82}$$

where we used $v^2 = T/\mu$.
Given our previous results

$$\alpha_n = \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right), \quad \text{and} \quad \beta_n = 0, \tag{83}$$

we thus find

$$\begin{aligned} E_{\text{tot}} &= \frac{L}{2}\sum_{n=1}^{\infty}\frac{\mu}{2}(\alpha_n^2 + \beta_n^2)\omega_n^2 \\ &= \frac{L}{2}\sum_{n=1}^{\infty}\frac{\mu}{2}\frac{16L^2}{(n\pi)^4}\sin^2\left(\frac{n\pi}{2}\right)\omega_n^2 \\ &= 4\mu L^3\sum_{n=1}^{\infty}\frac{1}{(n\pi)^4}\sin^2\left(\frac{n\pi}{2}\right)\left(\frac{n\pi v}{L}\right)^2 \\ &= 4\mu L v^2\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\sin^2\left(\frac{n\pi}{2}\right) \\ &= 4\mu L v^2\sum_{n=\text{odd\#s}}^{\infty}\frac{1}{n^2\pi^2}, \end{aligned} \tag{84}$$

where in the final line we used the fact that $\sin^2(n\pi/2) = 1$ for n odd and is zero otherwise. Equating this to result Eq.(82), we find

$$\frac{\mu v^2 L}{2} = 4\mu L v^2 \sum_{n=\text{odd\#s}}^{\infty}\frac{1}{n^2\pi^2}, \tag{85}$$

which when multiplied by $\pi^2/4\mu L v^2$ leads to the desired identity.

■