Solutions 5: Fourier Series and Wave Equations

Due Wednesday July 12, at 9AM under Rene García's door

Preface: In this assignment, we build a better understanding of Fourier Series and derive various wave equations.

1. **(15 points) Fourier Series identities**

We begin by computing the first integral. Using the given identity we find

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi}{L}x\right) + \cos\left(\frac{(n+m)\pi}{L}x\right)\right]\,dx \tag{1}$$

when $n = m$, we have

$$\begin{aligned}\int_0^L \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx &= \frac{1}{2}\int_0^L \left[1 + \cos\left(\frac{2m\pi}{L}x\right)\right]\,dx \\ &= \frac{L}{2} + \frac{L}{2m\pi}\sin\left(\frac{2m\pi}{L}x\right)\Big|_0^L = \frac{L}{2},\end{aligned} \tag{2}$$

where we used the fact that $\sin(2m\pi) = 0$ for all integers m.

For $n \neq m$, we have

$$\begin{aligned}\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx &= \frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi}{L}x\right) + \cos\left(\frac{(n+m)\pi}{L}x\right)\right]\,dx \\ &= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi}{L}x\right) + \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi}{L}x\right)\right]_0^L \\ &= 0.\end{aligned} \tag{3}$$

Thus we can infer

$$\boxed{\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \frac{L}{2}\delta_{nm}.} \tag{4}$$

Similarly, for the second integral we have

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \frac{1}{2}\int_0^L \left[\sin\left(\frac{(n-m)\pi}{L}x\right) + \sin\left(\frac{(n+m)\pi}{L}x\right)\right]\,dx \tag{5}$$

when $n = m$, we have

$$\begin{aligned}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx &= \frac{1}{2}\int_0^L \left[\sin\left(\frac{2m\pi}{L}x\right)\right]\,dx \\ &= \frac{L}{2m\pi}\cos\left(\frac{2m\pi}{L}x\right)\Big|_0^L = 0,\end{aligned} \tag{6}$$

where we used the fact that $\cos(2m\pi) = 1$ for all integers m.

For $n \neq m$, we have

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \frac{1}{2}\int_0^L \left[\sin\left(\frac{(n-m)\pi}{L}x\right) + \sin\left(\frac{(n+m)\pi}{L}x\right)\right]\,dx$$
$$= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\cos\left(\frac{(n-m)\pi}{L}x\right) + \frac{L}{(n+m)\pi}\cos\left(\frac{(n+m)\pi}{L}x\right)\right]_0^L, \tag{7}$$

Reducing the term in the brackets, we find

$$\frac{L}{(n-m)\pi}\cos\left(\frac{(n-m)\pi}{L}x\right) + \frac{L}{(n+m)\pi}\cos\left(\frac{(n+m)\pi}{L}x\right)$$
$$= \frac{L}{(n^2-m^2)\pi}\left[(n+m)\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) + (n+m)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\right.$$
$$\left. +(n-m)\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) - (n-m)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\right]$$
$$= \frac{2L}{(n^2-m^2)\pi}\left[n\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) + m\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\right], \tag{8}$$

which yields, upon re-substitution into Eq.(7),

$$\boxed{\int_0^L \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = \frac{nL}{(n^2-m^2)\pi}\Big[(-1)^{n+m}-1\Big]\Big[1-\delta_{nm}\Big]} \tag{9}$$

where we used $\sin(m\pi) = 0$ and $\cos(m\pi) = (-1)^m$ for all integers m. We note that the final expression contains the factor $[1-\delta_{nm}]$ which is only non-zero for $n \neq m$. Given the fact that $[(-1)^{n+m}-1]$ is itself only non-zero when $n \neq m$ (in particular when either n or m is odd), the factor $[1-\delta_{nm}]$ is redundant and can thus be excluded from the final expression.

Note: Eq.(9) is actually not one of the main identities of Fourier Series, but it is worth calculating to note that not all integrations of trigonometric functions over the domain 0 to L yield a simple Kronecker delta. The identity as it is typically quoted involves an integration over an even domain, yielding $\int_{-L/2}^{L/2} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx = 0$.

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2. **Summing Fourier Series**

   Soultion: Online

3. **Fourier Series with new Boundary Conditions**

   (a) **(10 points)** Applying separation of variables to the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2\frac{\partial^2 y(x,t)}{\partial x^2}, \tag{10}$$

   we found the general solution (with boundary conditions unspecified)

$$y(x,t) = \Big(A\cos(\omega t) + B\sin(\omega t)\Big)\Big(C\cos(kx) + D\sin(kx)\Big), \tag{11}$$

where $\omega = kv$. The boundary conditions for a string with free ends are

$$\frac{\partial}{\partial x} y(x=0,t) = 0, \qquad \frac{\partial}{\partial x} y(x=L,t) = 0. \tag{12}$$

Imposing these boundary conditions on Eq.(44) we find

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} y(x=0,t) \\ &= \Big(A\cos(\omega t) + B\sin(\omega t)\Big) Dk, \end{aligned} \tag{13}$$

implying $D = 0$, and

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} y(x=L,t) \\ &= -\Big(A\cos(\omega t) + B\sin(\omega t)\Big) Ck\sin(kL), \end{aligned} \tag{14}$$

implying $\sin(kL) = 0$. Thus we find that kL can be 0 or any positive integer n. We then have the wave number

$$k_n = \frac{n\pi}{L}, \quad \text{for } n = 0, 1, 2, \ldots. \tag{15}$$

We can have $n = 0$ in this case because $\cos(0) = 1$ yields a constant solution which is not necessarily zero and hence not trivial. Summing all possible solutions, given Eq.(**??**) and the value of D, we have

$$\begin{aligned} y(x,t) &= \sum_{n=0}^{\infty} y_n(x,t) \\ &= \sum_{n=0}^{\infty} \Big(A_n\cos(\omega t) + B_n\sin(\omega t)\Big) C_n\cos(k_n x) \\ &= \sum_{n=0}^{\infty} \Big[\alpha_n\cos(\omega_n t) + \beta_n\sin(\omega) nt\Big] \cos\left(\frac{n\pi}{L}x\right), \end{aligned} \tag{16}$$

where we redefined the arbitrary coefficients in our summation and took

$$\omega_n = \frac{n\pi v}{L}. \tag{17}$$

Separating the $n = 0$ term of the summation, we find

$$y(x,t) = \alpha_0 + \sum_{n=1}^{\infty} \Big[\alpha_n\cos(\omega t) + \beta_n\sin(\omega t)\Big] \cos\left(\frac{n\pi}{L}x\right). \tag{18}$$

The first term in Eq.(18) differs from the equation quoted in the prompt by a factor of $1/2$. This factor of $1/2$ is needed in order for the equation defining the coefficient for α_m when $m \geq 1$ to match the equation for α_m when $m = 0$. In anticipation of this result we will replace α_0 with $\alpha_0/2$, later checking that this replacement yields a value for α_m that is consistent for all m from 0 to ∞. Thus we postulate the conventional form of the solution

$$y(x,t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \Big[\alpha_n\cos(\omega t) + \beta_n\sin(\omega t)\Big] \cos\left(\frac{n\pi}{L}x\right). \tag{19}$$

(b) **(5 points)** With Eq.(19), we are now tasked with determining α_n and β_n. First, computing $y(x,0)$ and $\dot{y}(x,0)$ we have

$$y(x,0) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi}{L}x\right) \tag{20}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \beta_n \omega_n \cos\left(\frac{n\pi}{L}x\right). \tag{21}$$

Let's begin by determining α_0. Given Eq.(4), we know that integrating Eq.(20) from 0 to L would eliminate all the terms in the sum. So we have

$$\int_0^L y(x,0)\,dx = \int_0^L \frac{\alpha_0}{2}\,dx = \alpha_0 \frac{L}{2}. \tag{22}$$

For α_m, we can again use Eq.(20). Multiplying both sides of Eq.(20) by $\cos(m\pi x/L)$, and integrating from 0 to L, we obtain

$$\begin{aligned}
\int_0^L y(x,0)\cos\left(\frac{m\pi}{L}x\right)\,dx &= \int_0^L \left[\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi}{L}x\right)\right]\cos\left(\frac{m\pi}{L}x\right)\,dx \\
&= \sum_{n=1}^{\infty} \alpha_n \int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\,dx \\
&= \sum_{n=1}^{\infty} \alpha_n \frac{L}{2}\delta_{nm} = \alpha_m \frac{L}{2}, \qquad [\text{For } m = 1, 2, \ldots]
\end{aligned} \tag{23}$$

where we used Eq.(4) in the last two lines. Applying an identical procedure to Eq.(21), we obtain

$$\boxed{\beta_m = \frac{2}{L}\int_0^L \dot{y}(x,0)\cos\left(\frac{m\pi}{L}x\right)\,dx. \qquad [\text{For } m = 0, 1, \ldots]} \tag{24}$$

Consolidating Eq.(22) and Eq.(23), similarly yields

$$\boxed{\alpha_m = \frac{2}{L}\int_0^L y(x,0)\cos\left(\frac{m\pi}{L}x\right)\,dx. \qquad [\text{For } m = 0, 1, \ldots]} \tag{25}$$

Thus we see why the factor of $1/2$ was important in Eq.(19); it allowed us to combine the equations defining α_0 and $\alpha_{m\geq 1}$ into the single result Eq.(25).

(c) **(10 points)** Since the string begins from rest, we know that that $\dot{y}(x,0) = 0$ and thus by Eq.(24), $\beta_m = 0$. To determine α_n, we apply Eq.(25) given the $y(x,0)$ in the prompt. We find

$$\begin{aligned}
\alpha_m &= \frac{2}{L}\int_0^L y(x,0)\cos\left(\frac{m\pi}{L}x\right)\,dx \\
&= \frac{2}{L}\left[\frac{L}{2}\int_0^{L/2}\cos\left(\frac{m\pi}{L}x\right)\,dx - \frac{L}{2}\int_{L/2}^{L}\cos\left(\frac{m\pi}{L}x\right)\,dx\right] \\
&= \int_0^{L/2}\cos\left(\frac{m\pi}{L}x\right)\,dx + \int_{L/2}^{0}\cos\left(m\pi - \frac{m\pi}{L}u\right)\,du \\
&= \int_0^{L/2}\cos\left(\frac{m\pi}{L}x\right)\,dx - \cos(m\pi)\int_0^{L/2}\cos\left(\frac{m\pi}{L}u\right)\,du,
\end{aligned} \tag{26}$$

where in the second to last line we implemented the change of variables $u = L - x$. Simplifying this final line gives us

$$\begin{aligned}\alpha_m &= (1-\cos(m\pi))\int_0^{L/2}\cos\left(\frac{m\pi}{L}x\right)\,dx \\ &= (1-\cos(m\pi))\frac{L}{m\pi}\sin\left(\frac{m\pi}{L}x\right)\Big|_0^{L/2} \\ &= (1-\cos(m\pi))\frac{L}{m\pi}\sin\left(\frac{m\pi}{2}\right). \end{aligned} \tag{27}$$

For $m = 0$, Eq.(27) gives us $\alpha_0 = 0$. Also, for all even integers m, we have $1 - \cos(m\pi) = 0$, which is fine because $\sin(m\pi/2)$ is also zero for even m. But for m odd, we have $1 - \cos(m\pi) = 2$. Thus our final result for α_m is

$$\alpha_m = \frac{2L}{m\pi}\sin\left(\frac{m\pi}{2}\right), \tag{28}$$

and thus $y(x,t)$ is

$$\boxed{y(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos(\omega_n t)\cos\left(\frac{n\pi}{L}x\right),} \tag{29}$$

with $\omega_n = n\pi v/L$.

(d) **(Extra credit) (5 points)** Given the initial condition in the prompt, we know $y(0,0) = L/2$. Inserting $x = 0$ and $t = 0$ into Eq.(29), we find

$$\begin{aligned}\frac{L}{2} &= \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) \\ \frac{\pi}{4} &= \sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right) \\ &= \sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n}, \end{aligned} \tag{30}$$

where we used the property

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & \text{for } n = 1,5,9,\ldots \\ -1 & \text{for } n = 3,7,11,\ldots \\ 0 & \text{for } n = 2,4,6\ldots \end{cases} = (-1)^{\frac{n-1}{2}}, \tag{31}$$

in the final line.

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4. **String Wave Equation with Gravity**

(a) **(10 points)** Following the derivation in the notes, in the $\ell_R \ll a$ limit the mass j in the presence of a gravitational field has the equation of motion

$$m\ddot{y}_j = F_{j+1\text{ on }j,\,y} + F_{j-1\text{ on }j,\,y} = k(y_{j+1} - y_j) - k(y_j - y_{j-1}) - mg. \tag{32}$$

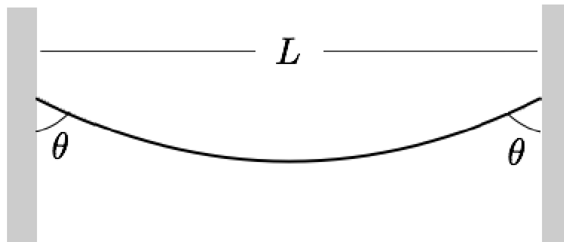

Figure 1: String in a gravitational field

Taking the continuum limit by promoting the $y_j(t)$ to $y(x,t)$, we have

$$m\ddot{y}(x,t) = k\left(y(x+a,t) - y(x,t)\right) - k\left(y(x,t) - y(x-a,t)\right) - mg. \tag{33}$$

Next, using the definitions

$$T = \lim_{a\to 0} ka, \quad \mu = \lim_{a\to 0} \frac{m}{a}. \tag{34}$$

we can divide Eq.(54) by a, taking the limit as $a \to 0$, to then find

$$\begin{aligned}
\lim_{a\to 0} \frac{m}{a}\, \ddot{y}(x,t) &= \lim_{a\to 0} \frac{k}{a}\left[\left(y(x+a,t) - y(x,t)\right) - \left(y(x,t) - y(x-a,t)\right)\right] - \lim_{a\to 0} \frac{m}{a} g \\
\mu\, \ddot{y}(x,t) &= \lim_{a\to 0} ka \frac{1}{a}\left[\frac{y(x+a,t) - y(x,t)}{a} - \frac{y(x,t) - y(x-a,t)}{a}\right] - \mu g \\
&= \lim_{a\to 0} ka \cdot \lim_{a\to 0} \frac{1}{a}\left[\frac{y(x+a,t) - y(x,t)}{a} - \frac{y(x,t) - y(x-a,t)}{a}\right] - \mu g \\
&= T y''(x,t) - \mu g
\end{aligned} \tag{35}$$

Replacing the dots with primes, we can write Eq.(55) as

$$\frac{\partial^2}{\partial t^2} y(x,t) = \frac{T}{\mu} \frac{\partial^2}{\partial x^2} y(x,t) - g, \tag{36}$$

which is equation of motion for a vibrating string in a gravitational field.

(b) **(5 points)**

To determine the configuration of a string in a gravitational field, we need to solve (**??**) in the time independent case. Specifically, we need to solve

$$\frac{d^2 y(x)}{dx^2} = \frac{\mu g}{T}. \tag{37}$$

subject to the appropriate boundary conditions. The general solution of Eq.(37) is

$$y(x) = y_0 + y_1 x + \frac{\mu g}{2T} x^2 \tag{38}$$

For a string attached to two walls the boundary conditions are set at each endpoint. The specific boundary conditions we need are

$$y(x=0) = y(x=L) = 0, \tag{39}$$

where L is the horizontal distance between the two endpoints. Thus, we find

$$y(0) = y_0 = 0, \qquad y(L) = y_1 L + \frac{\mu g}{2T} L^2, \tag{40}$$

which implies $y_1 = -\mu g L/2T$.

Inserting the above values into Eq.(38), we find that the specific solution to Eq.(37) is

$$\boxed{y(x) = \frac{\mu g}{2T}\, x(x - L).} \tag{41}$$

From this result we can make some simple consistency checks and calculate an interesting result. First, from symmetry we know that the slope of the string at each endpoint has the same magnitude. By calculating derivatives at each endpoint, we find this to be true of our model:

$$y'(x = 0) = -\frac{\mu g L}{2T} = -y'(x = L), \tag{42}$$

Also, we know that the string should have zero slope in its center. This is also found to be true:

$$y'(x = L/2) = 2\frac{\mu g}{2T}\frac{L}{2} - \frac{\mu g}{2T} L = 0. \tag{43}$$

We see our solution (37) passes the simple consistency checks.

As an aside, we can note that the general solution to Eq.(36) is a sum of the equilibrium solution (Eq.(41)) and the standard Fourier series solution we derived previously:

$$y(x,t) = \frac{\mu g}{2T}\, x(x - L) + \sum_{n=1}^{\infty} \Big[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\Big] \sin\left(\frac{n\pi}{L} x\right) \tag{44}$$

Thus, Eq.(44) represents wave motion on top of the parabolic curve

(c) **(5 points)** An interesting quantity we can calculate from this result is the angle the endpoints of the rope make with the vertical. This is computed easily using the derivative of the string and some simple trigonometry. Given the way we defined θ in the figure, we have

$$\tan\theta = \lim_{\Delta y} \frac{\Delta x}{\Delta y}\Big|_{x=0,L} = \frac{1}{y'(x)}\Big|_{x=0,L}. \tag{45}$$

Therefore, the angle the string makes with the vertical is given by

$$\boxed{\tan\theta = \frac{2T}{\mu L g} = \frac{2T}{Mg},} \tag{46}$$

where we defined $M \equiv \mu L$ as the total mass of the string. We note we could use this result (and a scale) to determine the tension in a hanging string, and, through $v = \sqrt{T/\mu}$, the speed of waves traveling through the string.

Eq.(46) reproduces the expected physical limits. For example, if we take the tension to infinity (the equivalent of pushing the walls very far apart) we find that the string is approximately a horizontal line, $\theta \simeq \frac{\pi}{2}$. If we make the string extremely loose and take the tension to zero (the equivalent of pushing the walls very close together) we find that the string roughly folds into two straight lines, $\theta = 0$.

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5. **Nonlinear Wave Equation**

(a) **(10 points)** Following the derivation outlined in the notes, we find that the magnitude of the force on the j oscillator from the $j+1$ oscillator is

$$|\vec{F}_{j+1 \text{ on } j}| = k\left(\sqrt{(y_{j+1}-y_j)^2 + a^2} - \ell_R\right). \tag{47}$$

Resolving this magnitude into its y component, the force in the y direction is

$$\begin{aligned} F_{j+1 \text{ on } j,\, y} &= |\vec{F}_{j+1 \text{ on } j}| \sin\theta_j \\ &= k\left(\sqrt{(y_{j+1}-y_j)^2 + a^2} - \ell_R\right) \cdot \frac{(y_{j+1}-y_j)}{\sqrt{(y_{j+1}-y_j)^2 + a^2}} \quad (48) \\ &= k(y_{j+1}-y_j)\left[1 - \frac{\ell_R}{a}\frac{1}{\sqrt{1+(y_{j+1}-y_j)^2/a^2}}\right], \quad (49) \end{aligned}$$

By a similar argument, we can compute the force on the jth oscillator from the $j-1$ oscillator. It is

$$\begin{aligned} F_{j-1 \text{ on } j,\, y} &= |\vec{F}_{j-1 \text{ on } j}| \sin\theta_{j-1} \\ &= -k\left(\sqrt{(y_j-y_{j-1})^2 + a^2} - \ell_R\right) \cdot \frac{(y_j-y_{j-1})}{\sqrt{(y_j-y_{j-1})^2 + a^2}} \quad (50) \\ &= -k(y_j-y_{j-1})\left[1 - \frac{\ell_R}{a}\frac{1}{\sqrt{1+(y_j-y_{j-1})^2/a^2}}\right], \quad (51) \end{aligned}$$

Thus the equation of motion for the jth oscillator is (by Newton's 2^{nd})

$$\begin{aligned} m\ddot{y}_j &= F_{j+1 \text{ on } j,\, y} + F_{j-1 \text{ on } j,\, y} \\ &= k(y_{j+1}-y_j)\left[1 - \frac{\ell_R}{a}\frac{1}{\sqrt{1+(y_{j+1}-y_j)^2/a^2}}\right] \\ &\quad - k(y_j-y_{j-1})\left[1 - \frac{\ell_R}{a}\frac{1}{\sqrt{1+(y_j-y_{j-1})^2/a^2}}\right]. \quad (52) \end{aligned}$$

Now, taking $\ell_R = a$ and $|y_j - y_{j-1}| \ll a$ for all j, we find

$$\left[1 - \left(1 + \frac{(y_j-y_{j-1})^2}{a^2}\right)^{-1/2}\right] = \left[1 - \left(1 - \frac{1}{2}\frac{(y_j-y_{j-1})^2}{a^2} + \cdots\right)\right] = \frac{1}{2}\frac{(y_j-y_{j-1})^2}{a^2} + \cdots \tag{53}$$

and so we can approximate the equation of motion as

$$m\ddot{y}_j = \lambda(y_{j+1}-y_j)\cdot\frac{1}{2}\frac{(y_{j+1}-y_j)^2}{a^2} - \lambda(y_j-y_{j-1})\cdot\frac{1}{2}\frac{(y_j-y_{j-1})^2}{a^2} + \cdots \tag{54}$$

Dropping the higher order terms while keeping the equality (as we would do in the standard derivation) gives us the equation of motion

$$m\ddot{y}_k = \frac{\lambda}{2a^2}\left[(y_{k+1}-y_k)^3 - (y_k-y_{k-1})^3\right]$$

$$= \frac{\lambda a}{2}\left[\left(\frac{y_{k+1}-y_k}{a}\right)^3 - \left(\frac{y_k - y_{k-1}}{a}\right)^3\right]$$

$$\frac{m}{a}\ddot{y}_k = \frac{\lambda a}{2}\frac{1}{a}\left[\left(\frac{y_{k+1}-y_k}{a}\right)^3 - \left(\frac{y_k - y_{k-1}}{a}\right)^3\right]. \tag{55}$$

Taking $a \to 0$, and defining $\mu \equiv m/a$ and $T \equiv \lambda a$, and taking $y_k(t) \to y(ka, t) = y(x, t)$ we obtain

$$\mu\frac{\partial^2 y}{\partial t^2} = \frac{T}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial y}{\partial x}\right)^3\right], \tag{56}$$

which clearly is a nonlinear wave equation for the string. To recap, the physical meaning of taking $\ell_R = a$ is to say that the string is loosely bound so that its rest state (and even perturbations about this rest state) have less tension than they do in the $\ell_R \ll a$ case.

(b) **(5 points)** Guessing the solution $y(x,t) = Ae^{i(kx-\omega t)}$ for Eq.(56), we find

$$\mu\frac{\partial^2 y}{\partial t^2} = \frac{T}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial y}{\partial x}\right)^3\right]$$

$$-\mu\omega^2 Ae^{i(kx-\omega t)} \stackrel{?}{=} \frac{T}{2}Ak^4 e^{3i(kx-\omega t)}. \tag{57}$$

which suggests that $y(x,t) = Ae^{i(kx-\omega t)}$ is not a solution. Indeed, since $y(x,t) = Ae^{i(kx-\omega t)}$ is a linear combination of sine and cosine functions, no sinusoidal solution would solve Eq.(56). We could have anticipated this from the fact that the equation is nonlinear and thus is not soluble through the "exponential guess and check" methods we previously employed.