Solutions 6: Electromagnetism and Final Exam Review Assignment

Problem 6 is due Sunday July 16, at 10PM under Rene García's door

Preface: The first part of this assignment consists of practice problems to review course material. The final problem provides an example of determining a magnetic field from an electric field. Use the first five problems as review (they are not to be handed in). **Only turn in Problem 6 on Sunday night.**

1. **Oscillations Near Equilibrium**
   A particle of mass m (confined to have position $x > 0$) is near the stable equilibrium of the potential

$$U(x) = \frac{\Delta_2}{x^2} - \frac{\Delta_4}{x^4}. \tag{1}$$

What are the units of Δ_2 and Δ_4? If the mass begins at rest a distance ℓ_0 away from the stable equilibrium, what is the speed of the particle when it passes the equilibrium position?

Solution:

Because $U(x)$ has units of Joules, Δ_2 and Δ_4 must have meters of J$\cdot$m^2 and J$\cdot$m^4, respectively.

Now, we want to find the speed of the particle as it passes through the equilibrium position given that it started a distance ℓ_0 away from that position. Let us take x_{eq} to be the equilibrium position. If the particle is a distance ℓ_0 away from its equilibrium position, then it is either at the position $x_{\text{eq}} + \ell_0$ or $x_{\text{eq}} - \ell_0$. By conservation of energy, the particle begins with an energy $U(x_{\text{eq}} + \ell_0)$ (or $U(x_{\text{eq}} - \ell_0)$) and when it passes the equilibrium position it has energy $U(\text{eq}) + \frac{1}{2}mv^2$. Thus we have the conservation of energy equation

$$U(x_{\text{eq}}) + \frac{1}{2}mv^2 = U(x_{\text{eq}} + \ell_0). \tag{2}$$

Solving for v^2 and employing the Taylor series approximation, we then have

$$\begin{aligned} v^2 &= \frac{2}{m}\left[U(x_{\text{eq}} + \ell_0) - U(x_{\text{eq}})\right] \\ &= \frac{2}{m}\left[U'(x_{\text{eq}})\ell_0 + \frac{1}{2}U''(x_{\text{eq}})\ell_0^2 + \mathcal{O}(\ell_0^3)\right]. \end{aligned} \tag{3}$$

By the definition of equilibrium, we have $U'(x_{\text{eq}}) = 0$. We therefore have

$$v = \sqrt{\frac{U''(x_{\text{eq}})}{m}\ell_0^2} \tag{4}$$

From Eq.(4) we note, that the speed is the same if we take $\ell_0 \to -\ell_0$, and thus to this order in the Taylor series it doesn't matter whether our particle began at $x_{\text{eq}} - \ell_0$ or $x_{\text{eq}} + \ell_0$.

Now, to use Eq.(4) to find the speed of the particle when it passes the stable equilibrium position, we need to find the stable equilibrium position first. Computing this equilibrium position, we have

$$U'(x) = 0 = -\frac{2\Delta_2}{x^3} + \frac{4\Delta_4}{x^5} = \frac{2}{x^3}\left(-\Delta_2 + \frac{2\Delta_4}{x^2}\right). \tag{5}$$

Thus, with $x > 0$, the equilibrium position is $x_{\text{eq}} = \sqrt{2\Delta_4/\Delta_2}$. Computing the second derivative of

$U(x)$ at this position we have

$$
\begin{aligned}
U''(x) &= \frac{6\Delta_2}{x_{\text{eq}}^4} - \frac{20\Delta_4}{x_{\text{eq}}^6} \\
&= 6\Delta_2 \left(\frac{\Delta_2}{2\Delta_4}\right)^2 - 20\Delta_4 \left(\frac{\Delta_2}{2\Delta_4}\right)^3 \\
&= -\Delta_2 \left(\frac{\Delta_2}{\Delta_4}\right)^2 ,
\end{aligned} \tag{6}
$$

which indicates that $\Delta_2 < 0$ and $\Delta_2 < 0$ in order for $x_{\text{eq}} = \sqrt{2\Delta_4/\Delta_2}$ to be a local minimum. Assuming[1] both of these conditions, we find that the velocity when the mass passes the equilibrium is

$$
\boxed{v = \sqrt{\frac{\ell_0^2}{m}|\Delta_2| \left(\frac{\Delta_2}{\Delta_4}\right)^2}.} \tag{7}
$$

Using our previous results identifying the units of Δ_2 and Δ_4, we can check that this result has the correct units. Since are computing a speed, the units of the quantity in the parentheses should be m^2/s^2. Checking this, we find

$$
\begin{aligned}
\left[\frac{\ell_0^2}{m}|\Delta_2| \left(\frac{\Delta_2}{\Delta_4}\right)^2\right] &= \left[\frac{\ell_0^2}{m}\right] \times \left[|\Delta_2|\right] \times \times \left[\left(\frac{\Delta_2}{\Delta_4}\right)^2\right] \\
&= \frac{\text{m}^2}{\text{kg}} \times \text{J} \cdot \text{m}^2 \times \frac{1}{\text{m}^4} \\
&= \frac{\text{J}}{\text{kg}} = \frac{\text{m}^2}{\text{s}^2},
\end{aligned} \tag{8}
$$

where we used the fact that $\text{J} = \text{kg} \cdot \text{m}^2/\text{s}^2$. Thus we see that Eq.(7) has the correct units.

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2. **Underdamped Oscillator**
An underdamped oscillator with phase $\phi = 0$ and initial amplitude A_0, starts off at the position $x(t = 0) = A_0$. The natural (i.e., undamped) frequency of the oscillator is ω_0 and the damping time constant is $\gamma = b/2m$ (with b the damping coefficient). At what time is the speed of the oscillator maximum? (Simplify result as much as possible)

Solution:

Our goal is to find the time at which an underdamped harmonic oscillator achieves maximum velocity. Given that we have an underdamped oscillator with phase $\phi = 0$ an initial amplitude A_0 and beginning at the position $x(t = 0) = A_0$, we find that the position of the oscillator as a function of time is

$$
x(t) = A_0 e^{-\gamma t} \cos(\Omega t), \tag{9}
$$

where $\Omega = \sqrt{\omega_0^2 - \gamma^2}$. We want to find the time at which the speed of this oscillator is maximum. Given the properties of the damped oscillator, this time should be such that as $\gamma \to 0$, $t_1 \to T/4 = \pi/2\omega_0$. To find this time, we set the acceleration of Eq.(9) to zero. First, computing the acceleration, we have

$$
\ddot{x}(t) = -A_0 e^{-\gamma t} \left[\Omega^2 \cos(\Omega t) - 2\gamma\Omega \sin(\Omega t) - \gamma^2 \cos(\Omega t)\right]. \tag{10}
$$

[1]Note to self: In a future problem, it would have been better to impose this criteria from the start

Setting this result to and noting that $e^{-\gamma t}$ can never be zero, we have

$$\tan(\Omega t) = \frac{\Omega^2 - \gamma^2}{2\gamma\Omega}, \tag{11}$$

or

$$\boxed{t = \frac{1}{\Omega}\tan^{-1}\left(\frac{\omega_0^2 - 2\gamma^2}{2\gamma\Omega}\right).} \tag{12}$$

As we expect, as we take $\gamma \to 0$, we have $\tan^{-1}(\cdots) = \tan^{-1}(\infty) = \pi/2$ and $t \to \pi/2\omega_0$.

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3. **Forced Oscillator**
A mass m is attached to a spring of spring constant k. The mass is at an equilibrium of the spring when it is at position $x = 0$. The mass begins from $x = 0$ with velocity v_0. Two forces $F_1(t)$ and $F_2(t)$ are applied to the mass as shown in Fig. 1. What is the position as a function of time $x(t)$?

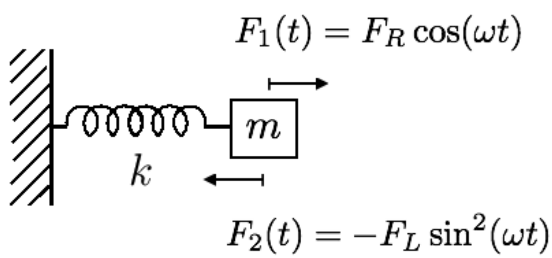

Figure 1

What should we get as $\omega \to 0$? What should we get as $F_L \to 0$?

Solution:

Our objective is to find $x(t)$ given the stated initial conditions. We will then consider how the system should change as $\omega \to 0$ and $F_L \to 0$. The equation of motion for the above system is

$$m\ddot{x} = -kx + F_R\cos(\omega t) - F_L\sin^2(\omega t). \tag{13}$$

Defining $\omega_0 = \sqrt{k/m}$ and using the identity $\sin^2 x = (1 - \cos(2x))/2$, we find

$$\ddot{x} + \omega_0^2 x = \frac{F_R}{m}\cos(\omega t) - \frac{F_L}{2m}\left(1 - \cos(2\omega t)\right), \tag{14}$$

defining

$$u \equiv x + \frac{F_L}{2m\omega_0^2}, \tag{15}$$

we find

$$\ddot{u} + \omega_0^2 u = \frac{F_R}{m}\cos(\omega t) + \frac{F_L}{2m}\cos(2\omega t) \tag{16}$$

Eq.(16) is a inhomogeneous linear differential equation and can thus be solved using the methods we discussed in Lecture notes 05. Doing so, we find the general solution

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_R}{m(\omega_0^2 - \omega^2)}\cos(\omega t) + \frac{F_L}{2m(\omega_0^2 - 4\omega^2)}\cos(2\omega t). \tag{17}$$

Thus, by Eq.(15) $x(t)$ is

$$x(t) = -\frac{F_L}{2m\omega_0^2} + A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_R}{m(\omega_0^2 - \omega^2)}\cos(\omega t) + \frac{F_L}{2m(\omega_0^2 - 4\omega^2)}\cos(2\omega t). \tag{18}$$

Imposing the initial conditions $x(t=0) = 0$ and $\dot{x}(t=0) = v_0$, we find the equations

$$-\frac{F_L}{2m\omega_0^2} + A + \frac{F_R}{m(\omega_0^2 - \omega^2)} + \frac{F_L}{2m(\omega_0^2 - 4\omega^2)} = 0 \tag{19}$$

$$B\omega_0 = v_0, \tag{20}$$

which yields, for $x(t)$,

$$\boxed{\begin{aligned} x(t) = {} & \frac{F_L}{2m\omega_0^2}\Big(\cos(\omega_0 t) - 1\Big) + \frac{v_0}{\omega_0}\sin(\omega_0 t) + \frac{F_R}{m(\omega_0^2 - \omega^2)}\Big(\cos(\omega t) - \cos(\omega_0 t)\Big) \\ & + \frac{F_L}{2m(\omega_0^2 - 4\omega^2)}\Big(\cos(2\omega t) - \cos(\omega_0 t)\Big). \end{aligned}}$$

As we take $\omega \to 0$, the force applied to the harmonic oscillator goes to the constant int time F_R. Thus the equation of motion reduces to something similar to that for a harmonic oscillator hanging vertically in a gravitational field. Thus, the only change in the standard solution of the harmonic oscillator should be a shift in the equilibrium value, which is what we find for Eq.(21):

$$\lim_{\omega \to 0} x(t) = \frac{F_R}{m\omega_0^2} + \frac{v_0}{\omega_0}\sin(\omega_0 t) - \frac{F_R}{m\omega_0^2}\cos(\omega_0 t). \tag{21}$$

Taking $F_L \to 0$, would reduce our system to one where only a single external force $F_R\cos(\omega t)$ is present. The solution would be similar to the case considered in Lecture notes 05, except now we have a non-zero initial velocity:

$$\lim_{F_L \to 0} x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t) + \frac{F_R}{m(\omega_0^2 - \omega^2)}\Big(\cos(\omega t) - \cos(\omega_0 t)\Big). \tag{22}$$

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4. **Coupled Oscillator**
Two identical springs and two identical masses are attached to a wall as shown in Fig. 2. Find the normal mode (angular) frequencies and the corresponding normal modes of the system.

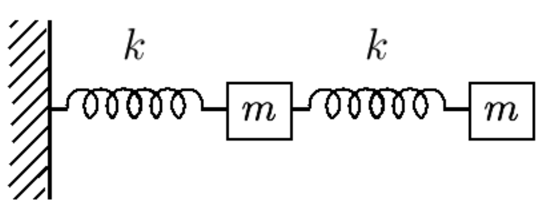

Figure 2

Solution:

We want to find the normal modes and normal mode frequencies (not necessarily in that order) for the system shown in Eq.(**??**). The equations of motion for the system are

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \tag{23}$$

$$m\ddot{x}_2 = -k(x_2 - x_1). \tag{24}$$

Writing these equations in matrix form, we have

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{25}$$

where we defined $\omega_0 = \sqrt{k/m}$. To find the normal mode frequencies we need to solve the eigenvalue-eigenvector equation

$$\begin{pmatrix} -2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} \tag{26}$$

and compute $|\mathrm{Im}[\alpha]|$ (absolute value is needed to ensure positive frequencies). We will then use our computed values of α^2 to determine the corresponding normal modes. Finding the characteristic equation, we have

$$\det \begin{pmatrix} -2\omega_0^2 - \alpha^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 - \alpha^2 \end{pmatrix} = \alpha^4 + 3\omega_0^2\alpha^2 + \omega_0^4 = 0, \tag{27}$$

which has the solutions

$$\alpha_\pm^2 = -\omega_0^2 \left(\frac{3 \pm \sqrt{5}}{2} \right). \tag{28}$$

Thus we find

$$\alpha_+ = i\omega_0 \left(\frac{3 + \sqrt{5}}{2} \right)^{1/2}, \qquad \alpha_- = i\omega_0 \left(\frac{3 - \sqrt{5}}{2} \right)^{1/2}, \tag{29}$$

where α_+ and α_- can also be equal to the negative of the stated values. Computing $|\mathrm{Im}[\alpha]|$, we find the normal mode frequencies

$$\boxed{\omega_+ = \omega_0 \left(\frac{3 + \sqrt{5}}{2} \right)^{1/2}, \qquad \omega_- = \omega_0 \left(\frac{3 - \sqrt{5}}{2} \right)^{1/2}.} \tag{30}$$

Now, to find the normal modes themselves. For α_+^2, we have the equation

$$\begin{pmatrix} -\omega_0^2(1 - \sqrt{5})/2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2(1 + \sqrt{5})/2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{31}$$

which yields

$$\begin{pmatrix} A \\ B \end{pmatrix} = c_+ \begin{pmatrix} 1 \\ (1 - \sqrt{5})/2 \end{pmatrix}. \tag{32}$$

Similarly, for α_-^2, we have the equation

$$\begin{pmatrix} -\omega_0^2(1 + \sqrt{5})/2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2(1 - \sqrt{5})/2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \tag{33}$$

which yields

$$\begin{pmatrix} A \\ B \end{pmatrix} = c_- \begin{pmatrix} 1 \\ (1+\sqrt{5})/2 \end{pmatrix}. \tag{34}$$

Therefore, we find that the normal modes of the system (corresponding to Eq.(30)) are

$$\boxed{\mathbf{v}_+ = \begin{pmatrix} 1 \\ (1-\sqrt{5})/2 \end{pmatrix}, \qquad \mathbf{v}_- = \begin{pmatrix} 1 \\ (1+\sqrt{5})/2 \end{pmatrix}.} \tag{35}$$

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5. **Fourier Series and Waves**

A vibrating string, of mass density μ and tension T, has fixed ends. The string is confined to be within a domain of length L and begins at $y(x,0) = 0$ for all possible x in the domain. However, the string also begins with a transverse velocity given by

$$\dot{y}(x,0) = v_0 \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) + v_0 \sin\left(\frac{3\pi x}{L}\right). \tag{36}$$

What is $y(x,t)$ at time $t = t_1$ where

$$t_1 = \frac{L}{3}\sqrt{\frac{\mu}{T}}, \tag{37}$$

written as a function of x? (Simplify result as much as possible)

Solution:

For the given initial conditions, we see $y(x,t)$ evaluated at $t = t_1$. First, we know that the solution to the wave equation for a wave with fixed ends is

$$y(x,t) = \sum_{n=1}^{\infty} \Big[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\Big] \sin\left(\frac{n\pi}{L}x\right), \tag{38}$$

where

$$\alpha_m = \frac{2}{L}\int_0^L dx\, y(x,0) \sin\left(\frac{m\pi}{L}x\right), \qquad \text{and} \qquad \beta_m = \frac{2}{L\omega_m}\int_0^L dx\, \dot{y}(x,0) \sin\left(\frac{m\pi}{L}x\right). \tag{39}$$

Since $y(x,0) = 0$, we find that $\alpha_m = 0$ for all m. Also, given the identity $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, we find

$$\dot{y}(x,0) = \frac{v_0}{2} \sin\left(\frac{4\pi x}{L}\right) + v_0 \sin\left(\frac{3\pi x}{L}\right), \tag{40}$$

Thus, we find for β_m

$$\begin{aligned} \beta_m &= \frac{2v_0}{L\omega_m}\int_0^L dx\, \sin\left(\frac{m\pi}{L}x\right) \left[\frac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \sin\left(\frac{3\pi x}{L}\right)\right] \\ &= \frac{v_0}{2\omega_m}\Big[\delta_{4m} + \delta_{3m}\Big]. \end{aligned} \tag{41}$$

Therefore $y(x,t)$ is

$$y(x,t) = \frac{v_0}{2\omega_4}\sin(\omega_4 t)\sin\left(\frac{4\pi x}{L}\right) + \frac{v_0}{\omega_3}\sin(\omega_3 t)\sin\left(\frac{3\pi x}{L}\right), \tag{42}$$

where $\omega_n = n\pi v/L$. Setting $t = L\sqrt{\mu/T}/3 = L/3v$, we find

$$\omega_3 t_1 = \frac{3\pi v}{L}\frac{L}{3v} = \pi \qquad \text{and} \qquad \omega_4 t_1 = \frac{4\pi v}{L}\frac{L}{3v} = \frac{4\pi}{3}. \tag{43}$$

Therefore, with $\sin(\pi = 0)$, we have

$$\boxed{y(x, t_1) = \frac{Lv_0}{8\pi v}\sin(4\pi/3)\sin\left(\frac{4\pi x}{L}\right) = -\frac{\sqrt{3}Lv_0}{16\pi v}\sin\left(\frac{4\pi x}{L}\right).} \tag{44}$$

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6. **Electromagnetism and Vector Calculus**
The electric field in a region of space is

$$\mathbf{E}(z,t) = E_0\Big(\cos(kz - \omega t)\,\hat{\mathbf{x}} + \sin(kz - \omega t)\,\hat{\mathbf{y}}\Big), \tag{45}$$

where E_0 has units of electric field, k is the wavenumber, and ω is the angular frequency.

(a) Using Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{46}$$

determine the magnetic field $\mathbf{B}(z,t)$ in this region of space. (Ignore any constants of integration)

(b) From your above results, compute $\mathbf{E} \cdot \mathbf{B}$.
(This shows that the electric and magnetic fields are perpendicular.)

Solution:

(a) We will use Faraday's Law to compute the magnetic field given the electric field in Eq.(45). Doing so we find

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0\cos(kz - \omega t) & E_0\sin(kz - \omega t) & 0 \end{vmatrix} = -E_0 k\cos(kz - \omega t)\,\hat{\mathbf{x}} - E_0 k\sin(kz - \omega t)\,\hat{\mathbf{y}}. \tag{47}$$

Setting this equal to the negative of the time-derivative of the magnetic field and integrating, we have

$$\begin{aligned} \mathbf{B}(z,t) &= \int dt\,\frac{\partial \mathbf{B}}{\partial t} \\ &= \int dt\,\Big[E_0 k\cos(kz - \omega t)\,\hat{\mathbf{x}} E_0 k\sin(kz - \omega t)\,\hat{\mathbf{y}}\Big] \\ &= \frac{E_0 k}{\omega}\Big(- \sin(kz - \omega t)\,\hat{\mathbf{x}} + \cos(kz - \omega t)\,\hat{\mathbf{y}}\Big), \end{aligned} \tag{48}$$

or with $k/\omega = 1/c$,

$$\boxed{\mathbf{B}(z,t) = \frac{E_0}{c}\Big(- \sin(kz - \omega t)\,\hat{\mathbf{x}} + \cos(kz - \omega t)\,\hat{\mathbf{y}}\Big)} \tag{49}$$

We note we could have also obtained this result by using $\mathbf{B} = \dfrac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}$ where $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, the direction of wave propagation.

(b) With Eq.(45) and Eq.(49), we find

$$\mathbf{E}(z,t)\cdot\mathbf{B}(z,t) = \frac{E_0^2}{c}\Big(-\cos(kz-\omega t)\sin(kz-\omega t) + \cos(kz-\omega t)\sin(kz-\omega t)\Big) = 0, \tag{50}$$

as expected for the electric and magnetic field parts of electromagnetic waves are perpendicular.

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