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2. **Simple Harmonic Motion**
   A particle undergoing simple harmonic motion has the position

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

This position can also be written as $x(t) = E\cos(\omega_0 t - \phi)$ where $E > 0$. What are E and ϕ in terms of A and B?

Solution: Using the sum of angles formula we find

$$\begin{aligned} x(t) &= E\cos(\omega_0 t - \phi) \\ &= E\cos\phi\cos(\omega_0 t) + E\sin\phi\sin(\omega_0 t). \end{aligned} \tag{1}$$

Equating this result to the given position, we find

$$E\cos\phi\cos(\omega_0 t) + E\sin\phi\sin(\omega_0 t) = A\cos(\omega_0 t) + B\sin(\omega_0 t), \tag{2}$$

which implies that $E\cos\phi = A$ and $E\sin\phi = B$. Using trigonometric identities and the condition $E > 0$, we then find

$$E = \sqrt{A^2 + B^2}, \qquad \phi = \tan^{-1}\frac{B}{A}. \tag{3}$$

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3. **Effective spring constant**
   Two springs with spring constants k_1 and k_2 are connected in parallel as shown in Fig. 1. What is the

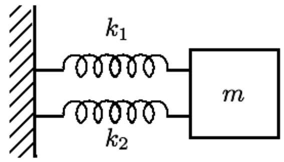

Figure 1

effective spring constant k_{eff}? In other words, if the mass is displaced by x, find the k_{eff} for which the force equals $F = -k_{\text{eff}} x$.

Solution: If the mass is displaced a distance x from its equilibrium position, the spring with spring constant k_1 exerts a restoring force $-k_1 x$. Similarly, the spring with spring constant k_2 exerts a restoring force $-k_2 x$ on the mass. Thus the net force on the mass is

$$F_{\text{net}} = -k_1 x - k_2 x = -(k_1 + k_2)x. \tag{4}$$

Therefore the effective spring constant is $k_{\text{eff}} = k_1 + k_2$.

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4. **Superposition**
Let $x_1(t)$ and $x_2(t)$ be solutions to the differential equation

$$\ddot{x}^2(t) = b\,x(t).$$

where b is a constant. Is the linear combination $x(t) = c_1x_1(t) + c_2x_2(t)$, where c_1 and c_2 are constants, a solution to the differential equation? (Provide a calculation showing why or why not).

Solution: If x_1 and x_2 solve the differential equation $\ddot{x}^2(t) = b\,x(t)$, then we have

$$\ddot{x}_1^2(t) = b\,x_1(t) \qquad \ddot{x}_2^2(t) = b\,x_2(t). \tag{5}$$

For $x(t) = c_1x_1(t)+c_2x_2(t)$ to be a solution to the differential equation, it too must satisfy the equations above. Checking whether it does, we have

$$\begin{aligned}
\left[\frac{d^2}{dt^2}\left(c_1x_1(t) + c_2x_2(t)\right)\right]^2 &\stackrel{?}{=} b\left[c_1x_1(t) + c_2x_2(t)\right] \\
\left(c_1\ddot{x}_1 + c_2\ddot{x}_2\right)^2 &\stackrel{?}{=} bc_1x_1 + bc_2x_2 \\
c_1^2\ddot{x}_1^2 + 2c_1c_2\ddot{x}_1\ddot{x}_2 + c_2^2\ddot{x}_2^2 &\stackrel{?}{=} bc_1x_1 + bc_2x_2 \\
c_1^2bx_1 + 2c_1c_2\ddot{x}_1\ddot{x}_2 + c_2x_2 &\neq bc_1x_1 + bc_2x_2 \qquad (6)
\end{aligned}$$

where in the last line we used Eq.(5). Thus the linear combination $c_1x_1(t) + c_2x_2(t)$ does not satisfy the differential equation and is not a solution to it.

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5. **An Anti-Hookean oscillator**
   A particle of mass m and at a position x is subject to a force

$$F(x) = +kx,$$

   with $k > 0$.

   (a) What is the most general form of $x(t)$ in terms of the parameters of the system?
   (b) The particle begins at x_0 and after long times (i.e., $t \to \infty$), the particle is at $x = 0$. What is the particle's initial velocity?

Solution: The equation of motion of this system is $m\ddot{x} - kx = 0$, or

$$\ddot{x} - \frac{k}{m}x = 0. \tag{7}$$

Finding the general solution by guessing a general exponential and fixing its parameters, we find

$$x(t) = Ae^{t\sqrt{k/m}} + Be^{-t\sqrt{k/m}}. \tag{8}$$

If the particle begins at x_0, we have $A + B = x_0$. If the particle is at $x = 0$ when $t \to \infty$, then we have $A = 0$. Thus, the specific solution is

$$x(t) = x_0 e^{-t\sqrt{k/m}}, \tag{9}$$

and the particle's initial velocity is $v_0 = -x_0\sqrt{k/m}$.

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6. **Exponential force**

   A particle of mass m is attached to a spring of spring constant k and is subject to another external force $F(t) = F_0 e^{-bt}$. The equation of motion of the system is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{-bt}, \tag{10}$$

   where $\omega_0^2 = k/m$. Guess a solution of the form $x(t) = Ae^{\alpha t}$ and determine what A and α should be in order for this solution to satisfy the above differential equation.

   Solution: Guessing a solution of the form $x(t) = Ae^{\alpha t}$ and plugging it into the equation of motion we find

$$(\alpha^2 + \omega_0^2) A e^{\alpha t} = \frac{F_0}{m} e^{-bt}. \tag{11}$$

   In order for this equation to be true for all time, we need to take $\alpha = -b$ and $A = F_0/[m(\alpha^2 + \omega_0^2)] = F_0/[m(b^2 + \omega_0^2)]$. We therefore find the particular solution to the given differential equation is

$$x(t) = \frac{F_0/m}{\omega_0^2 + b^2} e^{-bt}. \tag{12}$$

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7. **Matrix algebra**
   We have the following system of two equations of motion

$$m\ddot{x}_1 = -k_L x_1 + k_M(x_2 - x_1)$$
$$m\ddot{x}_2 = -k_R x_2 - k_M(x_2 - x_1),$$

where x_1 and x_2 are position variables and k_L, k_M, and k_R are spring constants. Say we want to write the two equations as a matrix equation where a 2×2 matrix multiplies a 2×1 matrix to yield a 2×1 matrix:

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

What should the values of the question marks be?

Solution: The equations of motion for x_1 and x_2 are

$$m\ddot{x}_1 = -(k_L + k_M)x_1 + k_M x_2$$
$$m\ddot{x}_2 = -(k_R + k_M)x_2 + k_M x_1,$$

Writing the above equation of motion as a matrix, we have

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -k_K - k_M & k_M \\ k_M & -k_R - k_M \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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8. **Second derivatives**

   Say we have the function

$$f(x) = x^2 + \sin(2x).$$

   Compute the value of

$$\lim_{a \to 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2}.$$

Solution: The given limit is the definition of a second derivative. Thus for the provided function we have

$$\lim_{a \to 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2} = \frac{d^2 f}{dx^2} = 2 - 4\sin(2x). \tag{13}$$

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9. **Introduction to Fourier Series**
Using the identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right],$$

compute the integral

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx$$

where n and m are both integers. Consider two cases: (i) $n = m$ and (ii) $n \neq m$

Solution: Using the given identity, we find

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx = \frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi}{L}x\right)\right]\,dx \tag{14}$$

when $n = m$, we have

$$\begin{aligned}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx &= \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2m\pi}{L}x\right)\right]\,dx \\ &= \frac{L}{2} + \frac{L}{2m\pi}\sin\left(\frac{2m\pi}{L}x\right)\Big|_0^L = \frac{L}{2}.\end{aligned} \tag{15}$$

For $n \neq m$, we have

$$\begin{aligned}\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\,dx &= \frac{1}{2}\int_0^L \left[\cos\left(\frac{(n-m)\pi}{L}x\right) - \cos\left(\frac{(n+m)\pi}{L}x\right)\right]\,dx \\ &= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi}{L}x\right) - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi}{L}x\right)\right]_0^L \\ &= 0.\end{aligned} \tag{16}$$

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10. **Solutions to Wave Equation**

We have the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} = v^2 \frac{\partial^2 y(x,t)}{\partial x^2},$$

where v is a parameter with units of velocity, and x and t are independent position and time variables, respectively. Which of the following functions could be solutions to this wave equation.

(a) $y(x,t) = h(x)g(vt)$

(b) $y(x,t) = g(x - vt)$

(c) $y(x,t) = h(x) + g(vt)$

(d) $y(x,t) = f(x + vt)$

Solution: (a), (b), (c), and (d) are all possible solutions to the wave equation. We can see this by guessing trial solutions.

For (a), $h(x) = Ae^{ikx}$ and $g(vt) = Be^{ikvt}$ yield $y(x,t) = Ce^{ik(x+vt)}$ (where $C = AB$) which is indeed a valid solution to the wave equation.

For (b), plugging in $g(x - vt)$ into the wave equation yields

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} y(x,t) &= v^2 \frac{\partial^2}{\partial x^2} y(x,t) \\
\frac{\partial^2}{\partial t^2} g(x - vt) &= v^2 \frac{\partial^2}{\partial x^2} g(x - vt) \\
(-v)^2 g(x - vt) &= v^2 g(x - vt) \\
g(x - vt) &= g(x - vt). \qquad (17)
\end{aligned}$$

Thus $g(x - vt)$ is a solution.

For (c), we find $y(x,t) = h(x) + g(vt)$ is a solution if $h(x)$ and $g(vt)$ satisfy

$$h(x) = h_0 + xh_1 + \frac{1}{2}x^2\delta \qquad (18)$$

$$g(vt) = g_0 + vtg_1 + \frac{1}{2}v^2t^2\delta, \qquad (19)$$

for arbitrary coefficients h_0, g_0, h_1, and g_1 and a constant δ all of which can be set with initial/boundary conditions.

For (d), we find $y(x,t) = f(x + vt)$ is a solution by a similar calculation to that in (b).

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11. **Electromagnetic Waves**
The electric field in an electromagnetic wave is given by

$$E_y(z,t) = E_0 \sin(kz - \omega t).$$

This electric field is related to the magnetic field, through the Maxwell equation

$$\frac{\partial B_x(z,t)}{\partial z} = \frac{1}{c^2}\frac{\partial E_y(z,t)}{\partial t},$$

where c is the speed of light. Assuming constants of integration are irrelevant, what is $B_x(z,t)$?

Solution: Differentiating the electric field with respect to time we have

$$\frac{\partial E_y(z,t)}{\partial t} = -\omega E_0 \cos(kz - \omega t). \tag{20}$$

Integrating, this result (and ignoring the constants of integration) with respect z, gives us

$$c^2 B_x(z,t) = \int dz\, \frac{\partial E_y(z,t)}{\partial t} = -\omega E_0 \int dz\ \cos(kz - \omega t) = -\frac{\omega}{k} E_0 \sin(kz - \omega t). \tag{21}$$

Thus

$$B_x(z,t) = -\frac{\omega}{kc^2} E_0 \sin(kz - \omega t) = -\frac{1}{c} E_0 \sin(kz - \omega t), \tag{22}$$

where we used $kc = \omega$.

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12. **The Predators and the Prey**
The following system of differential equations provides a very simple model for the number of rabbits R and the number of foxes F in a population:

$$\frac{dR}{dt} = \alpha R(t) - \beta R(t)F(t)$$
$$\frac{dF}{dt} = \delta R(t)F(t) - \gamma F(t),$$

where α, β, γ, and δ are constants. At what values of R and F do the number of rabbits and the number of foxes remain constant? (Find all such possible values)

Solution: For the number of rabbits and number of foxes to remain constant, we need $\dot{R}(t) = 0$ and $\dot{F}(t) = 0$. Thus

$$0 = \alpha R(t) - \beta R(t)F(t) = R(t)(\alpha - \beta F(t))$$
$$0 = \delta R(t)F(t) - \gamma F(t) = F(t)(\delta R(t) - \gamma), .$$

A trivial, time-independent solution is $R(t) = 0$ and $F(t) = 0$. The non-trivial solutions yielding constant R and F are

$$F(t) = \frac{\alpha}{\beta}, \qquad R(t) = \frac{\gamma}{\delta}. \tag{23}$$