Physics III: Final

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(Tuesday July 31, 9 AM)

First and Last Name: ________________________________

Exam Instructions:
This is an open-notebook exam, so feel free to use the notes you have transcribed throughout the summer and problem sets you have completed, but cellphones, laptops, and any notes written by someone else are prohibited. You will have **2 hours** to complete this exam.

Since this is a timed exam, your solutions need not be as "organized" as are your solutions to assignments. Short calculations and succinct explanations are acceptable, and you can state (without derivation) the standard results we derived in class. However, you should also recognize that you cannot receive partial credit for derivations/explanations you do not provide.

Challenge ________________

Problem 1 (10 pts): ________________

Problem 2 (30 pts): ________________

Problem 3 (20 pts): ________________

Problem 4 (20 pts): ________________

Problem 5 (25 pts): ________________

Problem 6 (25 pts): ________________

Total (130 pts):

1. **Protein Expression and Probability (20 points)**

   For a particular model of a gene in a cell, the probability density that said gene produces a concentration x of proteins during the cell cycle is given by

   $$p(x) = A \left(\frac{x}{b}\right)^N e^{-x/b}, \tag{1}$$

   where b is a biological constant with units of concentration, N is a physical constant, and A is a normalization parameter.

   (a) (5 points) The concentration of proteins that can be produced ranges from zero to infinite. What must A be in order for Eq.(1) to be normalized?

   (b) (5 points) What is the mean of the normalized probability density?

[Scratch work]

2. **Binary Alloy**[1] **(35 points)**

(a) $N = 8$ and $k = 0$ microstate

(b) Microstate with $N = 8$ and $k = 2$

Figure 2: Two different microstates for an $N = 8$ system. The figure on the left shows the original positions of the α and β atoms (i.e., the positions at temperature zero). The figure on the right shows a microstate in which $k = 2$ atoms (both α and β) have been displaced.

A binary alloy contains N identical atoms of type α and N identical atoms of type β. At low temperatures the system can be modeled as follows. There are N well-defined α-sites which are normally occupied by the α atoms and N well-defined β-sites which are normally occupied by the β atoms. At $T = 0$, the system is completely ordered, and all the α-sites are occupied by α atoms and same thing for β-sites and β atoms. However, at finite temperature, $k \leq N$ of the α atoms are displaced into the β-sites. (An equal number of β atoms is displaced to the α-sites).

The energy of the system is given by $E = \varepsilon k$, where ε is a constant with units of energy. To specify one of the many microstates of the system consistent with a particular value of k, one needs to indicate which α-sites are occupied by the k β atoms and which β-sites are occupied by the k α atoms. For example, if $k = 1$, then one α atom is displaced and there are N possible α atoms to choose from, and there are N possible β-site locations where it could be placed.

(a) (5 points) What is the number of different ways of choosing k of the α-sites to be vacated and occupied by β atoms?

(b) (5 points) What is the number of different ways of choosing k of the β-sites to be vacated and occupied by α atoms?

(c) (5 points) What is the free energy of the system as a function of k? *(Your answer can be written in terms of factorials or binomial coefficients)*

(d) (5 points) Take Stirling's approximation to be $\ln N! \simeq N \ln N - N$. What is the free energy of the system as a function of k after applying Stirling's approximation?

(e) (10 points) The system is in thermal equilibrium at a temperature T, and the number of displaced sites is $\overline{k}(T)$. Using the result from (c), and the properties of free energy at thermal equilibrium, determine $\overline{k}$ as a function of ε, N, and T.

[1] This problem is from an MIT 8.044 Open courseware exam.

[Scratch work]

[Scratch work]

3. **Model of Receptor Binding (20 points)**
On the cell membranes of cells, there are protein receptors to which extracellular molecules can bind and ultimately induce a signal in the cell. Let us consider a simple model of such receptor-molecule binding and analyze this model from the perspective of statistical physics.

Say we have many molecules each of which can either be free or bound to one of M distinct protein receptors (There are many more molecules than receptors). The molecules are identical to one another and each one has energy 0 when it is free and energy $-E_0$ when it is bound to a receptor. Our system exists at a temperature T. When the particles are free, we assume there is only *one* microstate for the free particles. An example microstate is shown in Fig. 3.

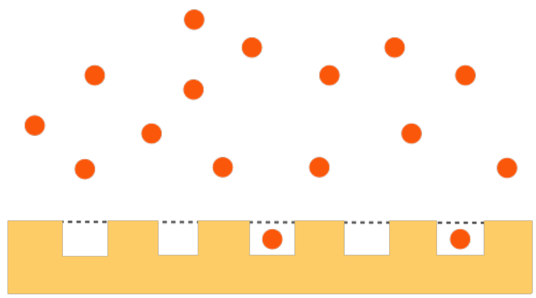

Figure 3: A particular microstate of a system with $M = 5$ receptor sites. There are two molecules bound to receptors so the energy of this microstate is $-2E_0$.

(a) (10 points) What is the partition function of the system written in terms of T, M, and E_0 (and a physical constant)? *(Your result should not have any unevaluated summations)*

(b) (5 points) Compute $\langle k \rangle$, the average number of molecules bound to receptors as a function of T.

(c) (5 points) At what temperature is an average of one molecule bound to the receptors?

[Scratch work]

4. **Ideal Gas of Distinguishable Particles (20 points)**
The partition function for an ideal gas of N distinguishable particles where particle k has mass m_k can be written as

$$Z = \frac{1}{h^{3N}} \int_V d^3\mathbf{q}_1 \int_{\text{all } \mathbf{p}} d^3\mathbf{p}_1 \cdots \int_V d^3\mathbf{q}_N \int_{\text{all } \mathbf{p}} d^3\mathbf{p}_N \, \exp\left(-\frac{1}{k_B T}\sum_{k=1}^{N} \frac{\mathbf{p}_k^2}{2m_k}\right).,\tag{2}$$

where V is the volume of the system.

(a) (10 points) Evaluate all the integrals in Eq.(2) and write the final result in terms of T, V, N and the set of masses $m_1, m_2, \ldots, m_N$ (and physical constants).

(b) (5 points) What is the free energy of this system written in terms of T, V, N and the set of masses $m_1, m_2, \ldots, m_N$ (and physical constants).

(c) (5 points) Given that pressure is the negative of the volume partial derivative of the free energy, derive the relationship between pressure P, number of particles N, volume V and temperature T for this system.

[Scratch work]

5. **Statistical Physics of Permutations (25 points)**
We have $2N$ objects consisting of N objects of type-B denoted $B_1, B_2, \ldots, B_N$ and N objects of type-W denoted $W_1, W_2, \ldots, W_N$. The objects can only exist in (B_k, W_ℓ) pairs, and the mircostates of our system are defined by a particular collection of pairings between the Bs and Ws. Fig. 4 depicts one such microstate for $N = 15$.

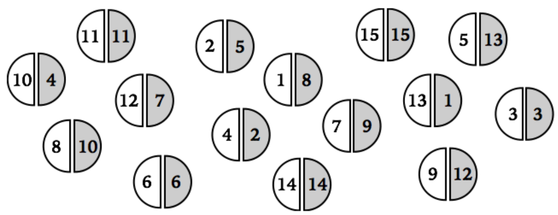

Figure 4: A particular microstate of a $N = 15$ system.

The energy of a microstate is the sum of the energies of all the pairs. The energy of a particular pair (consisting of (B_k, W_ℓ)) is

$$\mathcal{E}(B_k, W_\ell) = \begin{cases} 0 & \text{if } k = \ell, \\ \lambda & \text{if } k \neq \ell, \end{cases} \tag{3}$$

where $\lambda > 0$ is a parameter with units of energy. Namely, from Eq.(3), if a pair consists of (B_k, W_k), for any k, then the energy of the pair is zero, and if a pair consists of (B_ℓ, W_k), for $\ell \neq k$, then the energy of the pair is λ. We call the former a "matched pair" and the latter a "mismatched pair".

The partition function for this system can be written as

$$Z_N(\beta\lambda) = \sum_{j=0}^{N} g_N(j) e^{-\beta\lambda j} = \int_0^\infty dx\, e^{-x} \left[1 + (x-1)e^{-\beta\lambda}\right]^N, \tag{4}$$

where $\beta = 1/k_B T$ and where j is the number of mismatched pairs for a macrostate, and $g_N(j)$ is the number of microstates for a particular j. *You do not need to know the value of $g_N(j)$ to solve this problem*

(a) (5 points) Derive an expression for $\langle j \rangle$ in terms of the partition function and a partial derivative.

(b) (15 points) Use Laplace's method to evaluate the integral in Eq.(4)

(c) (5 points) Combining (a) and (b), what is $\langle j \rangle$ as a function of T?

[Scratch work]

6. **Transition Probabilities (25 points)**
We abstractly represent the connections between various states of a system with the picture below. In the picture, each filled circle represents a ”node”, and each line represents an ”edge.” When two nodes are linked by an edge, we say that they are ”connected.”

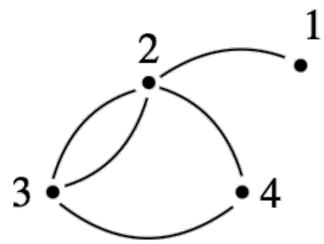

Figure 5

If at time t, we are at a node that is connected to M edges, then, in the next time step $t + \Delta t$, there is a probability of $1/M$ of traveling down any *one* of the connecting edges. There is a probability of zero of remaining at the same node.

(a) (5 points) Letting $\pi_{i\to j}$ represent the probability of transitioning from node i to node j in a single time step, fill in the elements below

$$\begin{array}{llll} \pi_{1\to 1} = & \pi_{1\to 2} = & \pi_{1\to 3} = & \pi_{1\to 4} = \\ \pi_{2\to 1} = & \pi_{2\to 2} = & \pi_{2\to 3} = & \pi_{2\to 4} = \\ \pi_{3\to 1} = & \pi_{3\to 2} = & \pi_{3\to 3} = & \pi_{3\to 4} = \\ \pi_{4\to 1} = & \pi_{4\to 2} = & \pi_{4\to 3} = & \pi_{4\to 4} = \end{array} \tag{5}$$

(b) (10 points) Say that at time t, the probability $p_j(t)$ to be at node j is given by

$$p_1(t) = \frac{1}{3}, \quad p_2(t) = \frac{1}{6}, \quad p_3(t) = \frac{1}{6}, \quad p_4(t) = \frac{1}{3}. \tag{6}$$

What is the probability to be at node 3 at time $t + \Delta t$?

(c) (10 points) We take $t \to \infty$. What is the probability to be at the various nodes?

[Scratch work]