Final Exam Information

Structure of Exam

- Open notebook exam (You can use any notes you write for yourself)
- 2 hour individual portion (consisting of 6-8 problems)

Final Exam Topics

Over the course of the summer we have studied eight main topics, all of which may be on the exam. The topics are listed below with their corresponding subtopics. Use the list to organize your review for the final.

1. Calculus, Probability, Combinatorics
   (a) Performing basic calculations in calculus (i.e., integration, differentiation, finding Taylor series)
   (b) Computing mean and standard deviation of discrete and continuous probability distributions
   (c) Using combinations and permutations to solve problems
2. Mathematical Definition of Information
   (a) Using definition to answer questions about information needed to specify microstate
3. Laws of Thermodynamics
   (a) Knowing the laws of thermodynamics and understanding their relevance to statistical physics
4. Free Energy and Order Parameters
   (a) Understanding difference between a microstate and a macrostate
   (b) Being able to compute the free energy of macrostate for discrete systems
   (c) Using the free energy to determine the temperature dependence of the macrostate at thermal equilibrium
5. Boltzmann Distribution and Partition Function
   (a) Being able to compute the partition function for systems given the definition of their microstates and the energy of a microstate
   (b) Using the partition function to compute macroscopic properties (i.e., average spin, average energy) of systems
6. Statistical Physics of the Ideal Gas
   (a) Understanding how the final form of the ideal gas partition function relates to the theoretical definition of partition function
   (b) Using partition function and results derived from partition function to solve problems
7. Laplace's Method

   (a) Being able to apply Laplace's method to approximate exponential integrals

8. Simulations in Statistical Physics

   (a) Understanding the definition of transition probabilities and how they determine time-dependent probabilities
   (b) Using detailed balance equation to determine equilibrium probabilities
   (c) Understanding the steps of Markov Chain Monte Carlo algorithm for simulating spin systems

Major Results

The below formula/methods represent major results for the associated topic. It might be useful to begin your exam study by reviewing the equations below and looking through the problem set/starter/practice problems associated with each one.

- **Mean of a function of a random variable**

$$\langle O(n)\rangle = \sum_n O(n)\, p_n \quad \text{[Discrete]}, \qquad \langle O(x)\rangle = \int_{-\infty}^{\infty} dx\, x\, p(x) \quad \text{[Continuous]} \tag{1}$$

- **Mean and Variance**

  For random variable n:

$$Mean = \langle n\rangle, \quad Variance = \langle n^2\rangle - \langle n\rangle^2 \tag{2}$$

- **Combinations and Permutations**

  Number of ways to create distinct groups of size $k \leq N$ when drawing from N distinct elements

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} \tag{3}$$

  Number of ways to order N distinct elements in an N-element list

$$N! \tag{4}$$

- **Mathematical definition of information**

  The quantity

$$I = -\sum_i p_i \log_2 p_i \tag{5}$$

  is the average number of binary-valued questions (under an optimal questioning strategy) needed to specify the state of a system. An 'optimal questioning strategy' refers to a series of questions which seek to divide the probability-weighted space of states in half after each question.

- **Entropy of a Macrostate (Boltzmann Entropy)**

$$S_N(M) = k_B \ln \Omega_N(M), \tag{6}$$

  where M denotes the macrostate, and $\Omega_N(M)$ is the number of microstates associated with M. $k_B = 1.3807 \times 10^{-23}$ J/K is Boltzmann's constant

- **Free Energy of a Macrostate**

$$F_N(M) = E_N(M) - TS_N(M), \tag{7}$$

where T is temperature, and $E_N(M)$ and $S_N(M)$ are the energy and entropy, respectively, of the macrostate M.

- **Partition Function of system at temperature T**

$$Z = \sum_{\{i\}} e^{-\beta E_i}, \tag{8}$$

where $\beta = 1/k_B T$, $\sum_{\{i\}}$ is the summation over all microstates, and E_i is the energy of microstate i

- **Ideal Gas Partition Function**

$$Z_{\text{ideal gas}} = \frac{V^N}{N!}\left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2}, \tag{9}$$

where where T is temperature, V is the volume of the system, N is the number of particles, and m is the mass of a single particle. $h = 6.626 \times 10^{-34}$ J·s is Planck's constant, and $k_B = 1.3807 \times 10^{-23}$ J/K is Boltzmann's constant.

- **Laplace's Method**

$$\int_B^A dx\, e^{-Nf(x)} \simeq \sqrt{\frac{2\pi}{Nf''(x_1)}}e^{-Nf(x_1)}, \tag{10}$$

where $x_1 \in [A, B]$, and x_1 is the point at which $f(x)$ is at a local minimum.

- **Transition and Equilibrium Probabilities**

$$\frac{\pi_{j\to k}}{\pi_{k\to j}} = \frac{p_k^{\text{eq}}}{p_j^{\text{eq}}}, \tag{11}$$

where $\pi_{j\to k}$ is the transition probability to move from microstate j to microstate k, and p_j^{eq} is the equilibrium probability to be in microstate k.

- **Definition of Factorial as an Integral**

$$N! = \int_0^\infty dx\, e^{-x} x^N \tag{12}$$

- **Hyperbolic Trigonometric Functions**

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{13}$$

- **Binomial Theorem**

$$(x+y)^N = \sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k} \tag{14}$$