Lecture 01: Introduction to Statistical Physics

In these notes, we define physics and discuss how the properties of physical theories suggest best practices for learning and applying them.

1 What is physics?

For the next six weeks, we will be studying statistical physics but before we begin, it will be useful to establish some groundwork. Beyond the ”statistical” what is ”physics” itself? When I posed this question to students from previous years I received many good answers. Here are a few of them

– Physics is the study of matter and energy

– Physics is the study of how the world works

– Physics is the study of the physical universe

All of these definitions are good and indeed correctly describe physics, but there is one definition which is both correct and is true to the ”spirit” of how physics is done:

> **Physics (definition):**
> Physics is what you get when you keep asking ”Why?”, ”What?” and ”How?” about the physical universe.

In one of your first physics classes, you likely studied a phenomena called ”projectile motion” in which an object is launched at a certain speed and angle of inclination through the air.

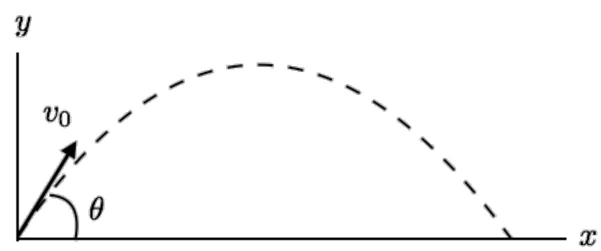


Figure 1: A ball is launched through the air at an initial speed v_0 and angle θ. One question an observer could ask is ”what path does the ball make in the air?”

This phenomena is a good first phenomena to study because most of us have thrown balls and watched as their paths made clean arcs before falling to the ground. A natural question (one consistent with the spirit of physics) is

”What path does the balls make after it is launched?”

The standard way of answering this question is to begin with Newton's second law for the ball as it moves in a gravitational field. The motion of the ball exists within a plane perpendicular to the surface of the earth, so we can parametrize the motion with the time-dependent coordinates $x(t)$ (denoting the horizontal position) and $y(t)$ (denoting the vertical position). For a ball of mass m the force in the y direction is $F_y = -mg$ and the force in the x direction is $F_x = 0$. Therefore, by Newton's second law we have

$$m\frac{d^2}{dt^2}x(t) = 0\,, \qquad m\frac{d^2}{dt^2}y(t) = -mg, \tag{1}$$

where $g = 9.81$ m/s^2 is the gravitational acceleration, and $d^2f(t)/dt^2$ denotes "the second-derivative with respect to time of $f(t)$." The equations in Eq.(1) are termed **differential equations**. A differential equation is generally any equation that includes the derivative of a function. Such equations are ubiquitous throughout physics (and generally any scientific discipline that uses calculus[1]).

For our system of interest in Fig. 1, we can answer the question of "What path does the balls make after it is launched?" by solving the equations in Eq.(1) for $x(t)$ and $y(t)$. By integrating both equations twice, we find the solutions

$$x(t) = x_0 + (v_0\cos\theta)t\,, \qquad y(t) = y_0 + (v_0\sin\theta)t - \frac{1}{2}gt^2, \tag{2}$$

where x_0 and y_0 are the initial horizontal and vertical positions, respectively, v_0 is the initial speed of the ball, and θ is the angle from the horizontal at which the ball is launched. Eq.(2) is a partial answer to our original question. It tells us how horizontal and vertical positions vary in time, but it does not exactly tell us the path that the particle makes as it moves through the air. To answer this latter question, we need to eliminate t from Eq.(2) and solve for y in terms of x. After some algebra, we obtain

$$y(x) = y_0 + (x - x_0)\tan\theta - \frac{g(x-x_0)^2}{2v_0^2\cos^2\theta}. \tag{3}$$

This is our answer! In Eq.(3), we see that the height is a quadratic function of the position and therefore thrown balls make parabolic trajectories in the air.

The work that led to Eq.(3) is a classical example of how classical mechanics is applied in practice. We come upon a question about particle motion; we formulate that question in the language of Newtonian mechanics using differential equations; we solve these differential equations and interpret the results in order to obtain a final answer to our original question.

One thing that should be noted about this system is that we are only studying the motion of one ball, or, more abstractly, one particle. This makes our analysis simpler because there is only a single differential equation to consider. However, not all systems of interest are so simple.

2 From the few to the many

Physics seeks to answer questions about the physical world, but the style of answers physics finds is very much dependent on the properties of the **systems**[2] being investigated. Classical mechanics is often the first subject physics students learn because the physical systems it studies are all around us. We are all subject to (and constantly aware of) the constant gravitational field of earth which makes particles move in parabolic trajectories. We intuitively know how acceleration relates to velocity and how velocity relates to position. And we know that when a ball hits a wall it will richohet off in a new direction which is theoretically calculable using classical mechanics.

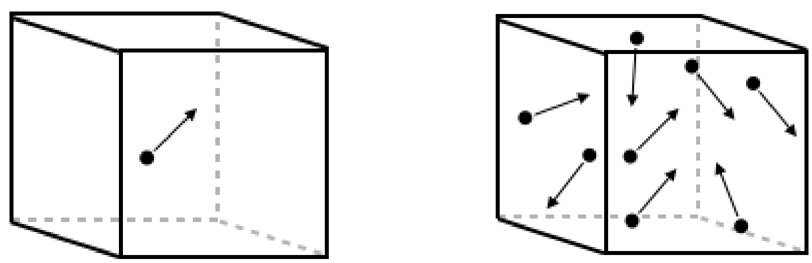
However, there are many systems all around us which seem to be comprehensible through classical mechanics, but which actually require different methods. For example, let's say we have a ball confined to a box. We assume there is no gravitational force in this system and that the walls of the box are perfectly elastic, so that the ball does not lose energy upon colliding with the walls. Given the ball's initial position and initial velocity, we could then use classical mechanics to determine the ball's velocity and position at any subsequent point in time. Any such real initial conditions invariably have some associated error. Namely,

[1] i.e., all scientific disciplines.

[2] "System" is a catchall term in physics for a specific context or problem we're studying. For example, a swing in a play ground, the gas molecules in a room, and the electromagnetic waves emanating from the sun all comprise systems.

we might not know the position and velocity of the particle exactly; only in some range of values. However, because we are only considering a single particle, the error that propagates from wall-collision to wall-collision is more easily recorded.

This is not the case when we have many particles in the box. When we have many (say $N \sim 100$) particles in the box, it becomes much more difficult to track all the particles' motion in time. Not only because there are more particles to keep track of but also because the errors in defining the initial positions and velocities of all the particles affect the outcome of each collision, and with $N \sim 100$ particles, there are so many collisions and so many errors in position and velocity that affect other errors in position and velocity that as time goes on it becomes increasingly difficult to say what position and what velocity any of the particles have. This phenomena is an example of **chaos**. Expressed succinctly, in all systems in classical physics, the present determines the future, but we describe a system as chaotic if the approximate present does not determine the approximate future.



(a) (b)

Figure 2: We can easily use classical mechanics to study the dynamics of a single particle in a box, but when there are many particles, initial errors in the trajectory of the particle

Classical mechanics is all about precisely specifying the future state of a system given information about its initial state. All information is approximate, and often such approximate information is good enough to get a fairly accurate idea of the future state. But when we have too many **degrees of freedom**, our predictions about the positions and velocities of the particles (or, more holistically, the specific trajectories of particles) are no longer accurate. Still, the goal of physics is to *accurately* describe the properties of physical systems, so if the specific trajectories we predict are no longer accurate, we need to find a new property to study.

With **statistical physics** we find that we may not be able to determine the *specific* state of a system with many degrees of freedom, but we can still often say something about the *probability* to be in various states. And by saying something about the probability to be in a certain state, we are able to compute averages of familiar quantities in classical physics like energy and velocity. This idea is the basic ethos of statistical physics: The time evolution of physical quantities for systems with many degrees is too difficult to study using the differential equations of classical physics, but by computing the probabilistic properties of these systems we can accurately compute time averages of physical quantities.

The idea of using probability and statistics (a mathematical subject primarily concerned with describing uncertainty) to study physics (a science subject seemingly concerned with precision) may seem contradictory. However, as we will see this summer, it is possible to make fairly precise predictions about systems which are inherently uncertain.

In moving from classical mechanics to statistical physics, we are moving away from a mathematical formalism that uses differential equations and toward a formalism that uses probabilities. The formal definition of the subject we will be studying during the summer is

Statistical Physics (definition):
Statistical physics is the foundational subject in physics which models the properties of systems with many degrees of freedom by using the methods of probability theory.

We can summarize the differing mathematical techniques in classical mechanics and statistical physics with a table.

Classical mechanics	Statistical physics
differential equations; solutions to differential equations; particle trajectories	probability theory; averages; combinatorics

Table 1: Mathematics used in classical mechanics and statistical physics

In statistical physics, we use our lack of precise information about the system to our advantage and use what we *do* know to define probability distributions. Thus, before we can begin our study of statistical physics, we would need to build up our understanding of probability. But even before we accumulate the necessary mathematical techniques, it will prove useful to prepare the intellectual landscape by reviewing some general features of a mathematical study of the physical world.

3 Features of Physics

As we build up the formalism of statistical physics, it is worth recognizing that the scientific discipline of physics bears certain properties that suggest best ways to learn it. Principally, physics **builds abstract models** of physical systems and **uses mathematics** both to make thinking more precise and to extrapolate patterns. When we combine abstraction and mathematics we get a particular type of mathematical modeling which for much of scientific history has been unique to physics

- **Abstraction:** This is the process of replacing concrete entities with more general, but often less physical representations. For example, if you were trying to get from Simmons to Lobby 13 on MIT's campus, you could abstract away the precise features of the terrain you are crossing by representing slightly curved streets by continuous lines and building's you don't enter as black boxes. The resulting representation would not be a perfectly faithful map of MIT's campus but it would bear features relevant to your travels.

  The utility of abstraction exists in the way it provides more general representations of a system, representations that can be more easily studied and more easily extended to new ones we may not have previously considered.

- **Mathematics:** For good reason, mathematics is said to be the language of physics. All of the major developments in theoretical physics, from Newtonian Mechanics in the 17th century to Quantum Mechanics in the 20th, have been expressed through formal and sometimes even new mathematics, and this mathematical expression lent rigor and precision to the claims these new theories made about the world.

  When we apply both abstraction and mathematics to the study of the physical world we obtain mathematical models that make the questions we ask about that world more precise. The combination also provides a framework for making extended logical arguments that can move us from the foundational assumptions of a subject to claims that can be compared with experiment.

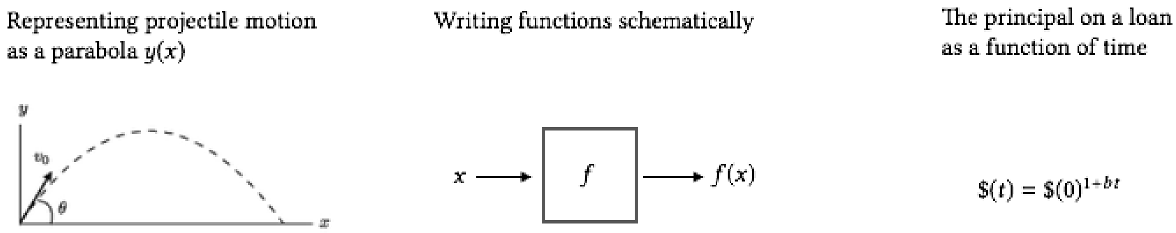


Figure 3: Three examples of the combination of abstraction with mathematics. Each abstraction provides a better way to understand the properties of the original system.

Concerning the combination of abstraction and mathematics, we should note that not all quantities that appear in our equations have clear physical counterparts. This probably was not the case when you were first studying classical mechanics. Time, positions, and forces–all quantities which are important in classical mechanics–have clear physical interpretations and can be directly measured. However, to a large extent, when studying statistical physics (and theoretical physics in general), we have to relinquish the reasonable desire to have a physical interpretation for every variable or function we define. Much of the formalism we build up will go towards solidifying the infrastructure of the theories we study, and although this infrastructure may not be physically transparent it will ultimately form the foundation for testable predictions about the world.

4 Doing and Learning Physics

Given the ways physics is built up (namely abstractly and with the use of mathematics), there are optimal and suboptimal practices for learning the subject. Physics is highly structured through principles and logical deductions and when studying physics it is helpful to make use of this structure. Doing so often entails limiting memorization only to the foundations of the subject, interrogating derivations so that they are properly understood, and trying to learn the connections between ideas in the subject.

- **Limiting Memorization:** Many academic subjects (like History or Anatomy) have an assortment of facts that must be memorized ever before the student can have a mature understanding of the subject. This is not the case in physics. In physics, there are some basic concepts and ideas students need to remember (e.g., conservation of energy and Newton's laws), but beyond these basics, understanding in physics is more dependent on being able to move from the basics of a subject to newer and deeper levels of physical and mathematical modeling.

  This course will have a lot of equations but it will be a poor strategy to try to memorize them all. Instead, you should work to understand how these equations are derived.

- **Internalizing Derivations:** In this class we will be working through many mathematical derivations. When one first sees a derivation the inclination might be to memorize each step in an effort to learn it. But we will be covering so many derivations that this will be a poor strategy. Moreover, it will make your later work more difficult because by simply memorizing the steps, you may not acquire the understanding needed to apply the result or to extend the derivation in new directions.

  So memorizing individual equations is a bad strategy for learning physics, and so too is trying to memorize the steps of a derivation that lead to these equations. A much more productive approach is to make note of three things when encountering a new derivation

  1. **What:** What question does the calculation seek to answer?

(a) (b)

Figure 4: Two types of understanding: (a) Taking ideas and equations as distinct and learning them as such. Leads to memorization and an inability to extend knowledge. (b) Understanding connections between ideas and equations. Allows for less memorization and develops skills to extend knowledge.

2. **How:** How is this question answered, namely what mathematical framework and techniques are needed to answer this question? If this mathematics is not familiar how can I learn it?
3. **Why:** Why is the starting question asked? What larger relevance does it have to the subject and to the world?

Being able to answer the above questions does not always mean one understands the derivation, but it does mean one has already acquired a deeper understanding than that obtained from memorization alone. See "Supplementary Notes 2" on the course website for a longer discussion of this idea.

- **Seeking Connections:** Physics is a somewhat unique academic discipline in that subject knowledge need not always be discerned from experiments, but can instead be mathematically derived. Due to this property, there is a structure to physics which makes it in many ways much easier to learn than other subjects without such structure. Every piece of knowledge in physics is logically and mathematically connected to every other piece of knowledge and so when learning the subject it is more efficient to seek to understand the connections between these pieces rather than to memorize the pieces themselves. See Fig. 4.

We can summarize these comments on best practices for learning and doing physics with an example. In some physics classes, the main results which define a subject are sometimes employed in a disconnected manner which suggests that these results were found, and hence should be applied, independently of one another. A standard example is "The Big Four" equations of kinematics which are often presented as distinct. However, these kinematic equation can all naturally be subsumed into the equation for constant acceleration:

$$\left.\begin{array}{l} x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \\ v(t) = v_0 + at \\ v_f^2 = v_0^2 + 2a(x_f - x_0) \\ x(t) = x_0 + \frac{1}{2}(v(t) + v_0) \end{array}\right\} \quad \rightarrow \quad \frac{d^2}{dt^2}x(t) = a \tag{4}$$

This is general in physics; all results in a physical theory (outside the physical principles and assumptions) are derived from and hence connected to other results. Thus in learning physics we should not memorize the results independent of one another, but should rather understand how they are connected. We depict this perspective in Fig. 4

Fig. 4 is meant to suggest that what you learn in a subject becomes more useful when you understand how the topics and ideas which define the subject are connected. One benefit of learning this way is that if you forget something (e.g., imagine if a node in Fig. 4 was erased), then if you understand how results in the subject are connected, you can rederive what you have forgotten from all that you still know.

To put it plainly: **only memorizing equations is an inefficient way to learn a subject which has as much manifest logical structure as physics does. Instead, efficiently learning physics entails learning the structure of the subject and how theorems and physical results are derived and are related to one another.**

Therefore, understanding in physics is not merely represented by what equations or ideas you know, but more truly in what you know about the connections between these equations and ideas. It is only by understanding these existing connections that you will ever have the knowledge and skills to move beyond them toward a comprehension of something entirely new.

5 What is next

Having now reviewed the best practices for learning physics, we can now turn towards learning one of its foundational subjects. We originally gave physics the informal definition of ”What you get when you keep asking ’Why?’ about the physical world.” Some such ”why?” questions are

- Why is the period of a pendulum independent of its mass?
- Why are certain elements non-reactive?
- Why does the pressure of a gas increase with its temperature?

In this class we will be concerned with questions of the last kind, namely questions dealing with a large number of particles or degrees of freedom. Such questions are the purview of statistical physics, and we will work to build the subject from the ground up, beginning with the mathematical definitions of entropy and the laws of thermodynamics and ultimately ending with applying the developed formalism to physical systems. But before we are ready to get into the physics, we must review the language (i.e., the mathematics) that its written in.