Assignment 1: Mathematics and Physics Review

Due Tuesday June 26 at 11:59PM under Fernando Rendon's door

Preface: Before you begin writing up your solution to this assignment, read **in full** the supplementary note 01 on "Presenting work". In particular pay attention to the "Write Legibly" and "Explanations are part of the solution" bullet points. In short, you should draft your work so that someone who is not accustomed to your hand writing can read it, and you should include both **qualitative** explanations and **mathematical** derivations in your solution.

1. **Introduce Yourself**
   Complete the survey at the link https://www.surveymonkey.com/r/RJ773VB.

2. **Sum and Product Notation**
   If we wanted a condensed way of writing the sum $a_1 + a_2 + \cdots + a_N$, we can use the sum notation defined by the Greek capital letter "sigma" (pronounced "syg-mah"), Σ:

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + \cdots + a_N. \tag{1}$$

   Below the summation symbol we have the equality "$i = 1$" indicating we begin our summation with i set to 1. The parameter "N" defines the final value of i in the summation. The quantity a_i is what we are summing.

   Similarly the product $a_1 \times a_2 \times \cdots \times a_N$ can be written in a condensed way using the product notation defined by the Greek capital letter "pi" (pronounced "pie"), Π:

$$\prod_{i=1}^{N} a_i = a_1 \times a_2 \times \cdots \times a_N. \tag{2}$$

   (a) Using the properties of exponentials write the following *product* as a *sum*

$$\prod_{i=1}^{N} e^{a_i} \tag{3}$$

   (b) Using the properties of logarithms write the following *sum* as a *product*

$$\sum_{i=1}^{N} \ln a_i \tag{4}$$

   (c) Using the properties of exponentials and logarithms compute the following integrals

$$\text{i. } \sum_{i=1}^{N} \int_1^{a_i} dx\, \frac{1}{x}\,, \qquad \text{ii. } \int_0^{\infty} dx \prod_{i=1}^{N} e^{-a_i x}. \tag{5}$$

   (Assume $\sum_{i=1}^{N} a_i > 0$ for the second integral.)

3. **Partial Derivative**
When we have the single-variable function $f(x)$, we know that the derivative of $f(x)$ can be denoted and defined as
$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{6}$$
Eq.(6) defines the instantaneous rate of change of $f(x)$ at the point x. If we have a two-variable function $f(x,y)$, we can similarly define a derivative along either the x or y directions. For example, the derivative along the x direction at the point (x,y) is denoted and defined as
$$\frac{\partial}{\partial x}f(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}. \tag{7}$$
The new symbol $\partial/\partial x$ is called "the partial derivative with respect to x" and it represents the instantaneous rate of change of the function $f(x,y)$ with respect to x at the point (x,y). We use a partial derivative $\partial/\partial x$ (rather than the *total* derivative d/dx) to point to the fact that we are differentiating a function of more than one variable. In practice, we compute Eq.(7), by simply computing the derivative with respect to x, while treating all other variables as if they were constants. For example, for the two-variable function
$$f_1(x,y) = \sin(xy^2)\,, \qquad f_2(x,y) = x^3y^2\,, \qquad f_3(x,y) = \ln(xy),$$
we can apply the chain rule to obtain
$$\frac{\partial}{\partial x}f_1(x,y) = y^2\cos(xy^2)\,, \qquad \frac{\partial}{\partial x}f_2(x,y) = 3x^2y^2\,, \qquad \frac{\partial}{\partial x}f_3(x,y) = \frac{1}{x}. \tag{8}$$
   (a) Using Eq.(7) as a model, denote and define "the partial derivative with respect to y" of $f(x,y)$ as a limit. What does this quantity represent?
   (b) Compute the partial derivative with respect to y of $f_1(x,y)$, $f_2(x,y)$, and $f_3(x,y)$ defined above.
   (c) For $f_1(x,y)$, compute
$$\frac{\partial^2}{\partial x\,\partial y}f(x,y) \equiv \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right) \qquad \text{and} \qquad \frac{\partial^2}{\partial y\,\partial x}f(x,y) \equiv \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}f(x,y)\right). \tag{9}$$
   How do each of these second-order derivatives relate to one another? What you have demonstrated is called **"the equality of mixed partials"** which states that for "well behaved"[1] multivariable functions, the result of higher-order partial differentiation is independent of the order in which you take the partial derivatives.

4. **Area of Projectile Motion**

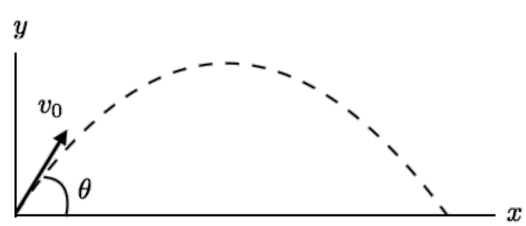

Figure 1: A ball is launched through the air at an initial speed v_0 and angle θ. In this problem, you will compute the area between the dotted line and the horizontal axis and the angle θ that maximizes this area.

[1]Which means functions which are differentiable at every point in their domain.

In a constant gravitational field, a ball is launched from height zero (i.e., $y = 0$) and position zero (i.e., $x = 0$) at an angle θ from the horizontal and at an initial speed v_0.

(a) What is the total horizontal distance the ball travels?

(b) What is $y(x)$, the height of the ball as a function of its position? Your answer should be in terms of g, v_0, and θ.

(c) The area under the curve of the ball's trajectory can be computed from

$$A(\theta) = \int_0^{X_{\text{tot}}} dx\, y(x), \tag{10}$$

where X_{tot} is the quantity computed in (a) and $y(x)$ is the trajectory computed in (b).
At what value of θ is $A(\theta)$ at a local maximum? *Hint: You should compute the integral first, and then apply the standard maximization algorithm in calculus.*