Assignment 3: Beginning Statistical Physics

Due Tuesday July 10 at 11:59PM under Fernando Rendon's door

Preface: In this assignment, we use the quantitative definition of information to make a bet in a number guessing game. We review the definition of microstate and macrostate. We derive the integral form of $N!$. And finally, we use the Boltzmann definition of entropy to study a simple model of a rubber band.

1. **Betting on Number Guessing**
   Your friend wants to play a round of the "Guess that number" game. She tells you that she has two decks of cards each of which is associated with the tens or the ones digit of a potential random number. For the tens-digit deck, **each of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8 occur once** and **the number 9 occurs eighteen times**. For the ones-digit deck, **each of the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8 occur once** and **the number 9 occurs nine times**.

   Your friend withdraws one card from each deck to represent a place value of a two digit number. She bets you some money that you will not be able to determine the hidden number in **five questions**. Should you take the bet? Explain why or why not.

2. **Personal Microstate and Macrostate Examples**
   We recall that when describing physical systems, "macrostates" are characterized by "macroscopic" (i.e., large scale) properties of a system such as total energy and total spin, and "microstates" are characterized by "microscopic" (i.e., small scale) properties of a system such as the energy or spin of a particular configuration.

   More formally, a **microstate** refers to a specific configuration of the system whereas a **macrostate** refers to the collection of configurations consistent with a certain condition (such as "all microstates with a total energy E"). For example, if we have a system of 3 lattice sites each of which could be 1 or 0, a macrostate could be having all lattice sites sum up to 2, and a particular microstate might be $\underline{1}\ \underline{0}\ \underline{1}$.

   For the following three systems, give one example of a macrostate that can describe the system as a whole and give one example of a microstate associated with that macrostate.

   (a) We flip a fair coin 10 times.

   (b) A bag is filled with 40 distinguishable balls equally divided between red, orange, yellow, and green colors (e.g., the balls are labeled $R_1, \ldots R_{10}, O_1 \ldots O_{10}, Y_1, \ldots, Y_{10}, G_1, \ldots, G_{10}$). We draw 10 balls from the bag.

   (c) We roll three fair six-sided dice.

3. **Gamma function**
   Although the factorial symbol (!) typically follows integers, there *is* a way to define it for any real[1] number.

   We define the function $\Gamma(N)$ (Greek capital letter gamma, pronounced "ga-mah") as

$$\Gamma(N) = \int_0^\infty dx\, e^{-x} x^{N-1}. \tag{1}$$

   (a) Using integration by parts, write $\Gamma(N+1)$ in terms of $\Gamma(N)$

   (b) What is $\Gamma(1)$?

   (c) Using (a) and (b), find a general formula for $\Gamma(N+1)$.

[1] The definition also applies to complex numbers but we will stick to real numbers in this course.

4. **Unusual Band**

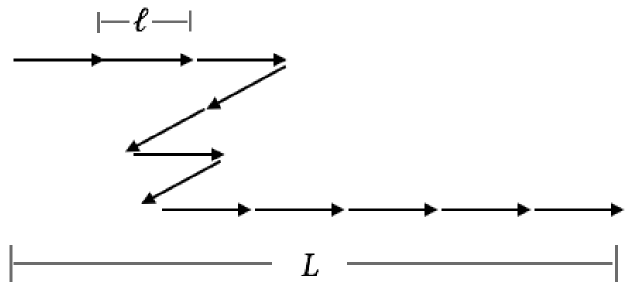

Figure 1: Rubber band

A one-dimensional model of a rubber band models the rubber band as a chain made up of N identical links of length ℓ. Counting the links starting from the left, each element can either be in the same direction or the opposite direction of the previous element. Therefore, each element either points to the left or to the right. We then have

$$N = n_+ + n_-, \qquad L = \ell(n_+ - n_-) = \ell(2n_+ - N), \tag{2}$$

where n_+ and n_- are the number of right-ward pointing links and the number of left-ward pointing links, respectively, and L is the total length of the chain.

(a) From Boltzmann, we know that when all the microstates of a system have the same energy and there are Ω microstates, the entropy of the system is

$$S = k_B \ln \Omega. \tag{3}$$

What is the entropy $S(N, n_+)$ of a system with n_+ rightward links and N total links? *Hint: Take all the microstates defined by particular N and n_+ to have the same energy.*
What is this entropy written exclusively in terms of L and N?

(b) Stirling's approximation (which we will later prove from Eq.(1)), states that for $N \gg 1$,

$$\ln N! = N \ln N - N + \frac{1}{2}\ln(2\pi N) + \mathcal{O}(N^{-1}). \tag{4}$$

For this problem, **we will neglect the $\frac{1}{2}\ln(2\pi N)$ term** in Eq.(4) because it is much less than the first term for $N \gg 1$. Using this approximation, rewrite the last result of (a) without using factorials.

(c) The force F of tension on the rubber band can be defined as

$$\frac{F}{T} = -\frac{\partial S}{\partial L}, \tag{5}$$

where T is temperature, S is the entropy, and L is the length of the rubber band. Compute the force of tension F as a function of the parameters of the system.

(d) Expand the result in (b) for small $L/N\ell$ to first order in L, and find the constant K in the expression

$$F \simeq KL \qquad \text{[For } L/N\ell \ll 1]. \tag{6}$$

If the force F is constant, and we heated up this rubber band, how would L change and why? *Also, watch this video:* ***Rubberband Thermodynamics***