Solutions 1: Physics and Mathematics Review

Due Tuesday June 26 at 11:59PM under XXXs door

Preface: Before you begin writing up your solution to this assignment, read **in full** the supplementary note 01 on "Presenting work". In particular pay attention to the "Write Legibly" and "Explanations are part of the solution" bullet points. In short, you should draft your work so that someone who is not accustomed to your hand writing can read it, and you should include both **qualitative** explanations and **mathematical** derivations in your solution.

1. **Introduce Yourself**
   Complete the survey at the link https://www.surveymonkey.com/r/RJ773VB➦.

2. **Sum and Product Notation**

   (a) (3 points) When we multiply two exponential functions, the result is an exponential with an argument equal to the sum of the arguments of each exponential factor in the product. Namely $e^{a_1}e^{a_2} = e^{a_1+a_2}$. Generalizing this result to the case of an N-term product, we have

$$\begin{aligned}\prod_{i=1}^{N} e^{a_i} &= e^{a_1}e^{a_2}\cdots e^{a_N}\\ &= e^{a_1+a_2+\cdots+a_N}\\ &= \exp\left(\sum_{i=1}^{N} a_i\right), \qquad (1)\end{aligned}$$

   where $\exp(x) \equiv e^x$. ■

   (b) (3 points) When we add two logarithmic functions (with the same base), the result is a logarithm with an argument equal to the product of the arguments of each factor in the sum. Namely $\ln a_1 + \ln a_2 = \ln a_1 a_2$. Generalizing this result to the case of an N-term sum, we have

$$\begin{aligned}\sum_{i=1}^{N} \ln a_i &= \ln a_1 + \ln a_2 + \cdots + \ln a_N\\ &= \ln a_1 a_2 \cdots a_N\\ &= \ln \prod_{i=1}^{N} a_i. \qquad (2)\end{aligned}$$

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   (c) i. (3 points) From single-variable calculus, we know that the derivative of the natural logarithm function $\ln x$ is $1/x$. Therefore, by the fundamental theorem of calculus, we have

$$\int dx\, \frac{1}{x} = \ln x + C. \qquad (3)$$

   For a fixed integration domain, from $x = 1$ to $x = a_i$, we have

$$\int_1^{a_i} dx\, \frac{1}{x} = \ln x\Big]_{x=1}^{x=a_i} = \ln a_i, \qquad (4)$$

where we used $\ln 1 = 0$. Summing this result from $i = 1$ to $i = N$, and using the result from (b), we obtain

$$\sum_{i=1}^{N} \int_1^{a_i} dx\, \frac{1}{x} = \ln \prod_{i=1}^{N} a_i. \tag{5}$$

ii. (3 points) For a parameter $b > 0$, we have

$$\int_0^\infty dx\, e^{-bx} = -\frac{1}{b} e^{-bx}\Big]_{x=0}^{x=\infty} = \frac{1}{b}. \tag{6}$$

Using the result in (a), we thus have the integral

$$\begin{aligned}\int_0^\infty dx\, \prod_{i=1}^{N} e^{-a_i x} &= \int_0^\infty dx\, \exp\left(-\sum_{i=1}^{N} a_i x\right) \\ &= \int_0^\infty dx\, \exp\left(-x \sum_{i=1}^{N} a_i\right) \\ &= \frac{1}{\sum_{i=1}^{N} a_i}, \end{aligned} \tag{7}$$

where we used Eq.(6) in the final line.

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3. **Partial Derivative**

(a) Using Eq.(**??**) as a model, denote and define "the partial derivative with respect to y" of $f(x, y)$ as a limit. What does this quantity represent.
Using the partial derivative in the prompt as a model, we can define the partial derivative of $f(x, y)$ with respect to y as

$$\frac{\partial}{\partial y} f(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}. \tag{8}$$

This quantity represents the instantaneous rate of change of the function $f(x, y)$ with respect to y at the point (x, y).

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(b) Compute the partial derivative with respect to y of $f_1(x, y)$, $f_2(x, y)$, and $f_3(x, y)$ defined above.
Given the model in the prompt we can compute the partial derivatives of each function by differentiating with respect to y and treating all other variables (namely, x) as constants. Doing so, we have, for the first function

$$\begin{aligned}\frac{\partial}{\partial y} f_1(x, y) &= \frac{\partial}{\partial y} \sin(xy^2) \\ &= 2xy \cos(xy^2). \end{aligned} \tag{9}$$

For the second function, we have

$$\begin{aligned}\frac{\partial}{\partial y} f_2(x, y) &= \frac{\partial}{\partial y} x^3 y^2 \\ &= 2x^3 y. \end{aligned} \tag{10}$$

And for the third function, we find

$$\frac{\partial}{\partial y} f_3(x,y) = \frac{\partial}{\partial y}\ln(xy)$$
$$= \frac{\partial}{\partial y}\left(\ln(x) + \ln(y)\right) = \frac{1}{y} \tag{11}$$

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(c) We will compute each mixed partial derivative as it is outlined in the prompt. Computing the first mixed partial derivative, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f_1(x,y)\right) = \frac{\partial}{\partial x}\left(2xy\cos(xy^2)\right)$$
$$= 2y\cos(xy^2) - 2y^3 x \sin(xy^2). \tag{12}$$

Computing the second mixed partial derivative, we have

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f_1(x,y)\right) = \frac{\partial}{\partial y}\left(y^2\cos(xy^2)\right)$$
$$= 2y\cos(xy^2) - 2y^3 x \sin(xy^2). \tag{13}$$

We see that Eq.(12) and Eq.(13) are equivalent. Thus, the order of the partial derivatives does not matter when computing the second partial derivative of this particular multivariable function. More generally, for functions which are continuously differentiable, the order of mixed partial derivatives does not affect the final result.

4. **Area of Projectile Motion**

(a) For a ball traveling in a constant gravitational, the time-dependent kinematical equations for the horizontal coordinate $x(t)$ and the vertical coordinate $y(t)$ are

$$x(t) = x_0 + v_{0x}t, \qquad y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2, \tag{14}$$

where x_0 and v_{0x} are the initial position and velocity in the x direction, and y_0 and v_{0y} are similarly defined. The acceleration due to gravity is $g = 9.81\ \text{m/s}^2$. For this system, we have $x_0 = y_0 = 0$ and $v_{0y} = v_0\sin\theta$ and $v_{0x} = v_0\cos\theta$. Therefore, the kinematic equations become

$$x(t) = (v_0\cos\theta)t \qquad y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2. \tag{15}$$

Our objective is to find the total horizontal distance the ball travels over its trajectory. We can find this horizontal distance by computing the time it takes to complete the trajectory and then inserting this time into the $x(t)$ equation in Eq.(15). From the figure, $y(t) = 0$ only at the start and the end of the projectile motion, so the total time for the trajectory can be determined by solving $y(t) = 0$ and ignoring the $t = 0$ solution. Doing so, using $y(t)$ in Eq.(15), we find the total time

$$t_{\text{tot}} = \frac{2v_0}{g}\sin\theta. \tag{16}$$

Inserting Eq.(16) into $x(t)$ in Eq.(15), gives us the total distance X_{tot}:

$$\boxed{X_{\text{tot}} = x(t_{\text{tot}}) = \frac{2v_0^2}{g}\cos\theta\sin\theta.} \tag{17}$$

We could use a trigonometric identity to further simplify Eq.(17), but we will leave it in its current form because this form will make subsequent calculations easier.

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(b) We can determine the height of the ball as a function of x, by eliminating t from the system of equations in Eq.(15). Solving for t in terms of x given the first equation in Eq.(15), we have

$$t = \frac{x}{v_0 \cos\theta}. \tag{18}$$

Substituting Eq.(18) into the $y(t)$ equation in Eq.(15), we have

$$y = v_0 \sin\theta \frac{x}{v_0 \cos\theta} - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\theta}, \tag{19}$$

or

$$\boxed{y(x) = \tan\theta\, x - \frac{gx^2}{2v_0^2 \cos^2\theta}.} \tag{20}$$

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(c) We seek to determine an angle θ at which the quantity

$$A(\theta) = \int_0^{X_{\text{tot}}} dx\, y(x), \tag{21}$$

is at a maximum. First we will use our previous results for $y(x)$ and X_{tot} to compute an explicit expression for $A(\theta)$. From (a) and (b), we find

$$\begin{aligned}
A(\theta) &= \int_0^{X_{\text{tot}}} dx\, y(x) \\
&= \int_0^{\frac{2v_0^2}{g}\cos\theta\sin\theta} dx \left[\tan\theta\, x - \frac{gx^2}{2v_0^2\cos^2\theta}\right] \\
&= \left[\tan\theta \frac{x^2}{2} - \frac{gx^3}{6v_0^2\cos^2\theta}\right]_0^{x=\frac{2v_0^2}{g}\cos\theta\sin\theta} \\
&= \frac{2v_0^4}{g^2}\tan\theta\cos^2\theta\sin^2\theta - \frac{4v_0^4}{3g^2}\frac{\cos^3\theta\sin^3\theta}{\cos^2\theta} \\
&= \frac{2v_0^4}{3g^2}\sin^3\theta\cos\theta.
\end{aligned} \tag{22}$$

Before we attempt to maximize this result, let us consider its basic dimensional and limiting case properties. We note that the expression in Eq.(22) has a factor of v_0^4/g^2 in the from. The units of this quantity are

$$\left[\frac{v_0^4}{g^2}\right] = \frac{(\text{m/s})^4}{(\text{m/s}^2)^2} = \frac{\text{m}^4 \cdot \text{s}^4}{\text{m}^2 \cdot \text{s}^4} = \text{m}^2, \tag{23}$$

indicating that our area has the correct units of meters-squared.

Also, as we take $\theta \to 0$ or $\theta \to \pi/2$, the total distance Eq.(17) traveled by the projectile goes to zero, because a ball launched horizontally from $y = 0$ and a ball launched vertically do not travel a horizontal distance. Therefore for $\theta \to 0$ and $\theta \to \pi/2$, we should expect the total area between the trajectory of the ball and the horizontal is zero; this is indeed what we find in Eq.(22).

Now, let's determine the value of θ that maximizes Eq.(22). We want to find θ_1 such that

$$A'(\theta = \theta_1) = 0, \qquad \text{and} \qquad A''(\theta = \theta_1) < 0, \tag{24}$$

for these define the conditions under which a function is at a local maximum. Differentiating Eq.(22) once and setting the result to zero, we find

$$A'(\theta = \theta_1) = \frac{2v_0^4}{3g^2}\left[3\sin^2\theta_1\cos^2\theta_1 - \sin^4\theta_1\right] = 0, \tag{25}$$

Dividing out $\sin^2\theta$ (which is non-zero because we know $\theta_1 = 0$ is not the sought for value) and ignoring the dimensional prefactor, we have the condition

$$0 = 3\cos^2\theta_1 - \sin^2\theta_1 = 3\cos^2\theta_1 - 1 + \cos^2\theta_1 = 4\cos^2\theta_1 - 1. \tag{26}$$

Solving for $\cos\theta_1$ then yields

$$\cos\theta_1 = \pm\frac{1}{2}. \tag{27}$$

From here on, we will choose the positive value of $\cos\theta_1$ because the image in the prompt does not depict the ball being thrown in the negative x direction. Therefore from Eq.(27), we can infer that $\theta_1 = \pi/3$ is a *possible* value at which $A(\theta)$ is maximized.

We have only shown that the derivative of $A(\theta)$ is zero at $\theta = \pi/3$. In order to show that $A(\theta)$ has a local maximum at this value, we need to show that the *second-derivative* is negative at this value of θ. Using Eq.(25) to compute the second derivative of $A(\theta)$ gives us

$$\begin{aligned} A''(\theta) &= \frac{2v_0^4}{3g^2}\left[6\sin\theta\cos^3\theta - 6\sin^3\theta\cos\theta - 4\sin^3\theta\cos\theta\right] \\ &= \frac{2v_0^4}{3g^2}\left[6\sin\theta\cos^3\theta - 10\sin^3\theta\cos\theta\right], \end{aligned} \tag{28}$$

Evaluating this function at $\theta = \pi/3$, and using $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/3) = 1/2$, we obtain

$$A''(\theta = \theta_1) = \frac{2v_0^4}{3g^2}\left[6\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{8} - 10\cdot\frac{3\sqrt{3}}{8}\cdot\frac{1}{2}\right] = \frac{2v_0^4}{3g^2}\left[-\frac{3\sqrt{3}}{2}\right] < 0. \tag{29}$$

Since $A''(\theta = \theta_1) < 0$ and $A'(\theta = \theta_1) = 0$, θ_1 is indeed a local maximum.

$\boxed{\text{Therefore, the function } A(\theta) \text{ is maximized at the angle } \theta = \pi/3.}$

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