Solutions 2: Probability and Counting

Due Tuesday July 3 at 11:59PM under Fernando Rendon's door

Preface: The basic methods of probability and counting (i.e., combinatorics) will prove to be crucial in our subsequent development of statistical physics. This problem set is meant to provide practice in these methods.

1. **Big $\mathcal{O}$ notation and Taylor series**

The problem states that we can look up the Taylor series for the three basic trigonometric functions and the Taylor series for the logarithmic function. The relevant Taylor series, taken from *Wolfram*'s *Mathworld*, are

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{1}$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2}$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots \tag{3}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \qquad \text{, for } |x| < 1 \cdots \tag{4}$$

where in the last equation we noted that the Taylor series for $\ln(1+x)$ only converges for $|x| < 1$.

(a) (3 points) To compute the Taylor series of $\sin(x)\cos(x)$, we can multiply Eq.(1) and Eq.(2), or use the identity

$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x) \tag{5}$$

with Eq.(1). Applying the identity, we have

$$\begin{aligned} \sin(x)\cos(x) &= \frac{1}{2}\sin(2x) \\ &= \frac{1}{2}\left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right] \\ &= x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots . \end{aligned} \tag{6}$$

There are three terms explicitly shown in Eq.(6). The next term in this expansion is some coefficient times x^7, so using Big $\mathcal{O}$ notation, we can write

$$\boxed{\sin(x)\cos(x) = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \mathcal{O}(x^7).} \tag{7}$$

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(b) (4 points) To compute the Taylor series of $\tan^2 x$ we can square Eq.(3). Doing, so we have

$$\tan^2(x) = \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots\right)^2$$

$$= x^2 + \frac{2}{3}x^4 + \frac{4}{15}x^6 + \frac{1}{9}x^6 + \cdots$$
$$= x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \cdots . \tag{8}$$

There are three terms explicitly shown in Eq.(6). The next term in this expansion is some coefficient times x^8, so using Big $\mathcal{O}$ notation, we can write

$$\boxed{\tan^2 x = x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \mathcal{O}(x^8)} \tag{9}$$

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(c) (3 points) To compute the Taylor series of $\ln(1+x^2)$, we can use Eq.(4) and substitute x^2 for x. Doing so, we have

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots . \tag{10}$$

Beyond the first three terms, the next term is of order x^8, so we can write this result in Big $\mathcal{O}$ notation as

$$\boxed{\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \mathcal{O}(x^8).} \tag{11}$$

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2. **Number of Hands of Poker**

- (5 points) **Straight Flush:** A poker hand containing cards of sequential rank and all of the same suit. The rank $\mathbb{A}$ can act as both the low rank and the high rank so that both $\mathbb{A}$♥ 2♥ 3♥ 4♥ 5♥ and 10♦ $\mathbb{J}$♦ $\mathbb{Q}$♦ $\mathbb{K}$♦ $\mathbb{A}$♦ are straight flushes.

  To have a straight flush, we must have all five cards of the same suit and of sequential rank. Because we have five cards in each hand, the cards that can act as the starting rank in such a hand are the 10 cards $\mathbb{A}, 2, 3, 4, 5, 6, 7, 8, 9, 10$. We cannot have a straight flush where the lowest starting rank is higher than rank 10 because in such a hand the lowest starting rank will actually be higher than at least one card. Thus there are 10 ranks we can start from. Since there are 4 suits, there are $10 \times 4 = 40$ cards which can function as starting cards. After we select a starting card, for each subsequent card there is only 1 choice for the card of the same suit and next highest rank. So, we have

$$40 \times 1 \times 1 \times 1 \times 1 = 40, \tag{12}$$

  possible straight flushes. We do not divide by any factorial because the order *is* important in defining a straight.

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- (5 points) **Full House:** A poker hand containing three cards of one rank and two cards of another rank. *Example:* $\mathbb{A}$♥ $\mathbb{A}$♠ $\mathbb{A}$♦ 8♥ 8♣.

  To have a full house, we must have three cards of one rank and two cards of a second rank. Any card can be chosen to be in the three cards of the same rank or two cards of the same rank set. For the three-card set, there are 52 cards to choose from to select the first card of this set. After this first choice, we only have 3 other cards of the same rank but different suit. After a second choice of one of these 3, we have 2 other cards of the same rank but different suit to complete the three-card set.

To create the two-card set, we have to ignore all the cards of the rank used to compose the three-card set. This leaves us with $52 - 4 = 48$ cards to choose from as the first card of the two-card set. After this first choice, we only have 3 other cards of the same rank but different suit to choose from to complete the two card set.

Thus there are $52 \times 3 \times 2 \times 48 \times 3$ ways to select a *particular* ordering of three cards of one rank and a *particular* ordering of two-cards of a second rank. However, the ordering amongst the cards in the three-card set and amongst the cards in the two-card set is not important. Therefore, we correct our result by dividing by $3! \times 2!$ representing the product of the number of equivalent ways to reorder the three-card set multiplied by the number of equivalent ways to reorder the two-card set. We thus have

$$\frac{52 \times 3 \times 2 \times 48 \times 3}{3! \times 2!} = 3,744, \tag{13}$$

possible full houses.

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- (5 points) **Flush:** A poker hand containing five cards all of the same suit but *not* of sequential rank (i.e., not a straight flush). *Example:* 4♠ 6♠ 𝕂♠ 9♠ 2♠.

  To create a flush, we need five cards all of the same suit, but *not* all of sequential rank. To compute the number of straights, we will compute the number of five-card combinations with the same suit, regardless of whether the ranks are sequential, and then subtract the number of straight flushes.

  In selecting the first card for a flush, we can choose any card from our deck. Thus there are 52 possible choices for the first card. After this first choice, there are $12 \times 11 \times 10 \times 9$ ways to choose the four remaining cards of the same suit. Thus, there are $52 \times 12 \times 11 \times 10 \times 9$ ways to select a particular ordering of five cards with the same suit. However, order in a flush does not matter, so we have to divide this result by $5!$. Subtracting 40 (i.e., the number of straight flushes) from the result of this division, we have that the number of flushes is

$$\frac{52 \times 12 \times 11 \times 10 \times 9}{5!} - 40 = 5,108 \tag{14}$$

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- (5 points) **Straight:** A poker hand containing five cards of sequential rank but *not* all of the same suit (i.e., not a straight flush). *Example:* 2♥ 3♠ 4♦ 5♥ 6♣.

  To create a straight, we need five cards of sequential rank but *not* all of the same suit. To compute the number of straights, we will compute the number of five card combinations of cards with sequential ranks, regardless of suit, and then subtract the number of straight flushes.

  As we argued in the Straight Flush part, there are 40 choices for the first card of a straight. After this choice, we have 4 cards (one rank across the four suits) to choose from in selecting the next card. This is true (i.e., we have 4 cards to choose from) for all subsequent cards until we complete the hand. Thus there are $40 \times 4 \times 4 \times 4 \times 4$ possible straights we can create (regardless of suit). We do not divide this number by a factorial because order is important in defining a straight. Now, knowing the number of possible straight flushes (i.e., 40 of them), we find the number of straights (as defined in the prompt) by subtracting the number of straight flushes from the total number of straights regardless of suit. We thus find that the number of straights is

$$40 \times 4 \times 4 \times 4 \times 4 - 40 = 10,200. \tag{15}$$

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3. **Mean and variance for various distributions**

(a) For an unfair coin with a probability p of getting heads and a probability of $1-p$ of getting tails (in one flip), the probability of getting $n \leq N$ heads in N flips is

$$\text{Prob}(n) = \binom{N}{n} p^n (1-p)^{N-n}, \tag{16}$$

where n can run from $n=0$ to $n=N$. We note that Eq.(16) satisfies the standard normalization requirement.

$$1 = \sum_{n=0}^{N} \binom{N}{n} p^n (1-p)^{N-n} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}. \tag{17}$$

Computing the mean of the number of flips, we have

$$\begin{aligned}
\sum_{n=0}^{N} n\, \text{Prob}(n) &= \sum_{n=0}^{N} n \frac{N!}{n!(N-n)!}\, p^n (1-p)^{N-n} \\
&= \sum_{n=1}^{N} \frac{N!}{(n-1)!(N-n)!}\, p^n (1-p)^{N-n} \\
&= Np \sum_{n=1}^{N} \frac{(N-1)!}{(n-1)!(N-n)!}\, p^{n-1} (1-p)^{N-n} \\
&= Np \sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-n)!}\, p^k (1-p)^{N-1-k}.
\end{aligned} \tag{18}$$

In the second line, we started our summation from $n=1$ because the $n=0$ term yielded zero. In the third line, we factored Np from the expression. In the final line, we redefined our summation label as $k = n-1$ in order to put the result in a more useful form. Using the identity in Eq.(17), we can take the summation in the final line to be 1. We thus find that the mean of the binomial distribution is

$$\boxed{\langle n \rangle = Np.} \tag{19}$$

This result matches our intuition. If we have an unfair coin for which p is the probability of getting heads after one coin flip, we expect there to be about Np heads after N coin flips.

In order to compute the variance, we must first compute $\langle n^2 \rangle$. Taking the advice of the prompt, we will begin by computing $\langle n(n-1) \rangle$. The resulting calculation is similar to that used to compute $\langle n \rangle$. Computing the result, we have

$$\begin{aligned}
\langle n(n-1) \rangle &= \sum_{n=0}^{N} n(n-1) \frac{N!}{n!(N-n)!}\, p^n (1-p)^{N-n} \\
&= \sum_{n=2}^{N} \frac{N!}{(n-2)!(N-n)!}\, p^n (1-p)^{N-n} \\
&= N(N-1)p^2 \sum_{n=2}^{N} \frac{(N-2)!}{(n-2)!(N-n)!}\, p^{n-2} (1-p)^{N-n} \\
&= N(N-1)p^2 \sum_{\ell=0}^{N-2} \frac{(N-2)!}{\ell!(N-n)!}\, p^{\ell} (1-p)^{N-2-\ell}.
\end{aligned} \tag{20}$$

In the second line, we started our summation from $n=2$ because the $n=0$ and $n=1$ terms yielded zero. In the third line, we factored $N(N-1)$ from the expression. In the final line, we redefined our summation label $\ell = n-2$ in order to put the result in a more useful form. Using the identity in Eq.(17), we can take the summation in the final line above to be 1. We thus find

$$\langle n(n-1)\rangle = \langle n^2\rangle - \langle n\rangle = N(N-1)p^2. \tag{21}$$

Or $\langle n^2\rangle = \langle n\rangle + N(N-1)p^2 = Np + N(N-1)p^2$. Computing the variance, we have

$$\begin{aligned}\sigma_n^2 &= \langle n^2\rangle - \langle n\rangle^2 \\ &= Np + N(N-1)p^2 - N^2p^2 \\ &= Np - Np^2,\end{aligned} \tag{22}$$

or

$$\boxed{\sigma_n^2 = Np(1-p).} \tag{23}$$

This value of the variance matches our intuitive understanding of variance. If $p = 0$ or $p = 1$, there is no spread in our distribution (all of the probability is concentrated in one of two values), and so the variance should be (and is) zero.

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4. **Gaussian integral**

(a) Our goal is to compute the integral

$$\int_{-\infty}^{\infty} dx \, \exp\left(-\alpha x^2 + \beta x + \gamma\right) \tag{24}$$

in terms of c_0, where c_0 is defined by

$$\int_{-\infty}^{\infty} dx \, \exp\left(-x^2\right) = c_0. \tag{25}$$

We begin by completing the square in the argument of the integrand of Eq.(24). We have

$$\begin{aligned}-\alpha x^2 + \beta x + \gamma &= -\alpha\left(x^2 - \frac{\beta}{\alpha}x\right) + \gamma \\ &= -\alpha\left(x^2 - 2\frac{\beta}{2\alpha}x - \frac{\beta^2}{4\alpha^2} + \frac{\beta^2}{4\alpha^2}\right) + \gamma \\ &= -\alpha\left(x - \frac{\beta}{2\alpha}\right)^2 + \frac{\beta^2}{4\alpha} + \gamma.\end{aligned} \tag{26}$$

Using the properties of the exponential, we then have

$$\begin{aligned}\exp\left(-\alpha x^2 + \beta x + \gamma\right) &= \exp\left(-\alpha\left(x - \frac{\beta}{2\alpha}\right)^2 + \frac{\beta^2}{4\alpha} + \gamma\right) \\ &= \exp\left(-\alpha\left(x - \frac{\beta}{2\alpha}\right)^2\right)\exp\left(\frac{\beta^2}{4\alpha} + \gamma\right).\end{aligned} \tag{27}$$

Returning to the integral, we have

$$\int_{-\infty}^{\infty} dx\, \exp\left(-\alpha x^2 + \beta x + \gamma\right) = \exp\left(\frac{\beta^2}{4\alpha} + \gamma\right) \int_{-\infty}^{\infty} dx\, \exp\left(-\alpha\left(x - \frac{\beta}{2\alpha}\right)^2\right) \tag{28}$$

where we factored the x-independent exponential from the integral. Using u-substitution we can then define

$$u \equiv \sqrt{\alpha}\left(x - \frac{\beta}{2\alpha}\right). \tag{29}$$

From this definition, we have $du = \sqrt{\alpha}\, dx$ (or $dx = du/\sqrt{\alpha}$), $u_0 = -\infty$ and $u_f = \infty$. The integral then becomes

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, \exp\left(-\alpha x^2 + \beta x + \gamma\right) &= \exp\left(\frac{\beta^2}{4\alpha} + \gamma\right) \int_{-\infty}^{\infty} du\, \frac{1}{\sqrt{\alpha}} \exp(-u^2) \\ &= \exp\left(\frac{\beta^2}{4\alpha} + \gamma\right) \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} du\, \exp(-u^2) \\ &= \boxed{\exp\left(\frac{\beta^2}{4\alpha} + \gamma\right) \frac{c_0}{\sqrt{\alpha}},} \end{aligned} \tag{30}$$

which is the desired result in terms of c_0.

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(b) We now turn to a calculation of

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2}. \tag{31}$$

Squaring the result and moving to polar coordinates, we have

$$I^2 = \int_0^{2\pi} d\phi \int_0^{\infty} dr\, r\, \exp\left(-r^2\right). \tag{32}$$

The integrand is ϕ independent, so we can perform the ϕ integral right away. Doing so gives us

$$I^2 = 2\pi \int_0^{\infty} dr\, r\, \exp\left(-r^2\right) = \pi \int_0^{\infty} dr\, 2r\, \exp\left(-r^2\right). \tag{33}$$

For the final step, we will use u-substitution and define $u = r^2$. This gives us $du = 2r\, dr$, $u_0 = 0$, and $u_f = \infty$. The integral then becomes

$$I^2 = \pi \int_0^{\infty} du\, e^{-u} = \pi. \tag{34}$$

where we used $\int_0^{\infty} du\, e^{-u} = -e^{-u}|_0^{\infty} = 1$. Taking the square root of Eq.(34) and choosing the positive root (because I is the integral of an exclusively positive function), we have

$$\boxed{I = \sqrt{\pi}.} \tag{35}$$

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(c) Combining the results from (a) and (b), we find

$$\boxed{\int_{-\infty}^{\infty} dx\, \exp\left(-\alpha x^2 + \beta x + \gamma\right) = \exp\left(\frac{\beta^2}{4\alpha} + \gamma\right) \sqrt{\frac{\pi}{\alpha}}.} \tag{36}$$

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5. **The 68, 95, 99.7 Rule**

(a) Our probability density is

$$p(x) = \frac{e^{-(x-x_0)^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}}. \tag{37}$$

By the definition of probability density, the probability to be in a domain of x values between $x_0 - k\sigma_0$ and $x_0 + k\sigma_0$ is the definite integral of $p(x)$ within this domain. Namely,

$$\boxed{\text{Prob}(x_0 - k\sigma_0 \leq x \leq x_0 + k\sigma_0) = \int_{x_0-k\sigma_0}^{x_0+k\sigma_0} dx\, \frac{e^{-(x-x_0)^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}}.} \tag{38}$$

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(b) We will reduce the result in Eq.(38) so that it is only a function of k. We define

$$u = \frac{x - x_0}{\sigma_0\sqrt{2}}, \tag{39}$$

which gives us $du = dx/\sigma_0\sqrt{2}$ (or $dx = \sigma_0\sqrt{2}\,du$) and

$$u_0 = \frac{x_0 - k\sigma_0 - x_0}{\sigma_0\sqrt{2}} = -\frac{k}{\sqrt{2}}, \tag{40}$$

and

$$u_f = \frac{x_0 + k\sigma_0 - x_0}{\sigma_0\sqrt{2}} = +\frac{k}{\sqrt{2}}. \tag{41}$$

Eq.(38) then becomes

$$\begin{aligned} \text{Prob}(x_0 - k\sigma_0 \leq x \leq x_0 + k\sigma_0) &= \int_{-k/\sqrt{2}}^{k/\sqrt{2}} du\, \sqrt{2}\sigma_0\, \frac{e^{-u^2}}{\sqrt{2\pi\sigma_0^2}} \\ &= \frac{1}{\sqrt{\pi}} \int_{-k/\sqrt{2}}^{k/\sqrt{2}} du\, e^{-u^2}. \end{aligned} \tag{42}$$

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(c) Using WolframAlpha, we find

i.

$$\begin{aligned} \text{Prob}(x_0 - \sigma_0 \leq x \leq x_0 + \sigma_0) &= \frac{1}{\sqrt{\pi}} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} du\, e^{-u^2} \\ &= 0.6827 \end{aligned} \tag{43}$$

ii.

$$\begin{aligned} \text{Prob}(x_0 - 2\sigma_0 \leq x \leq x_0 + 2\sigma_0) &= \frac{1}{\sqrt{\pi}} \int_{-2/\sqrt{2}}^{2/\sqrt{2}} du\, e^{-u^2} \\ &= 0.9545 \end{aligned} \tag{44}$$

iii.

$$\text{Prob}(x_0 - 3\sigma_0 \leq x \leq x_0 + 3\sigma_0) = \frac{1}{\sqrt{\pi}} \int_{-3/\sqrt{2}}^{3/\sqrt{2}} du\, e^{-u^2}$$
$$= 0.9973 \tag{45}$$

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(d) The quantities for parts i., ii., and iii. of (c) give, respectively, the probability to find the random variable one-standard deviation, two-standard deviations, and three-standard deviations away from the mean.

The values in (c) are often summarized as the $68-95-99.7$ rule. They are important in probability theory because the probability distribution Eq.(37) occurs in many areas of statistics (also occurs in statistical physics!), and thus it is useful to know how the cumulative probability changes as we expand our interval of consideration around the mean.

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