Solution 3: Beginning Statistical Physics

Due Tuesday July 10 at 11:59PM under Fernando Rendon's door

Preface: In this assignment, we use the quantitative definition of information to make a bet in a number guessing game. We review the definition of microstate and macrostate. We derive the integral form of $N!$. And finally, we use the Boltzmann definition of entropy to study a simple model of a rubber band.

1. **Betting on Number Guessing**

   (10 points) Our goal is to determine whether we should take our friend's bet, and the determining factor will be whether we can guess the hidden number in five questions or fewer, on average. If we determine that we can guess the hidden number in five or fewer questions, on average, then we will take the bet. Otherwise, we will note take the bet.

   In Lecture Notes 03 "Entropy and Information", we determined that if there is a probability p_i of getting a number i out of a set of numbers $\{i\}$, then the average number of questions we will need to guess the number (under the given conditions of the "Guess that number" game) is

$$\langle \text{\# of Qs} \rangle = -\sum_{\{i\}} p_i \log_2 p_i. \tag{1}$$

   For the game as it is outlined in the prompt, we only have the number from $i = 0$ to 99 inclusive, so Eq.(1) becomes

$$\langle \text{\# of Qs} \rangle = -\sum_{i=0}^{99} p_i \log_2 p_i. \tag{2}$$

   From the prompt, we know that the hundreds digit has eighteen 9-cards and one card each of $0, 1, 2, 3, 4, 5, 6, 7, 8$. Therefore, the probability of getting a 9 in the hundreds digit is $18/(18+9) = 2/3$, and the probability of getting any other *particular* number in the hundreds digit is $1/3 \times 1/9 = 1/27$.

   Similarly, since the tens digit has nine 9-cards and one card each of $0, 1, 2, 3, 4, 5, 6, 7, 8$, the probability of getting a 9 in the tens digit is $9/(9+9) = 1/2$, and the probability of getting any other *particular* number in the tens digit is $1/2 \times 1/9 = 1/18$.

   From these probabilities, we can determine the probability to get any number between 0 and 99. We can divide these numbers into four categories (and four associated probabilities) contingent on whether there is a 9 in one of the digits:

$$p_{99} = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3} \tag{3}$$

$$p_{9B} = \frac{2}{3} \times \frac{1}{18} = \frac{1}{27} \qquad \text{[Where } B \neq 9 \text{ .]} \tag{4}$$

$$p_{A9} = \frac{1}{27} \times \frac{1}{2} = \frac{1}{54} \qquad \text{[Where } B \neq 9\text{.]} \tag{5}$$

$$p_{AB} = \frac{1}{27} \times \frac{1}{18} = \frac{1}{486} \qquad \text{[Where } A \neq 9 \text{ and } B \neq 9\text{.]} \tag{6}$$

   We can then write Eq.(2) as

$$\langle \text{\# of Qs} \rangle = -p_{99} \log_2 p_{99} - \sum_{\{9B\}} p_{9B} \log_2 p_{9B} - \sum_{\{A9\}} p_{A9} \log_2 p_{A9} - \sum_{\{AB\}} p_{AB} \log_2 p_{AB}, \tag{7}$$

where the summations are over all numbers of the associated form. Given that there are nine numbers of the form $9B$ (where $B \neq 9$) and nine numbers of the form $A9$ (where $A \neq 9$) and $100-1-9-9 = 81$ numbers of the form AB where ($A \neq 9$ and $B \neq 9$), we find that Eq.(7) becomes

$$\begin{aligned}\langle \text{\# of Qs} \rangle &= -p_{99}\log_2 p_{99} - 9\times p_{9B}\log_2 p_{9B} - 9\times p_{A9}\log_2 p_{A9} - 81\times p_{AB}\log_2 p_{AB} \\ &= -\frac{1}{3}\log_2\frac{1}{3} - 9\times\frac{1}{27}\log_2\frac{1}{27} - 9\times\frac{1}{54}\log_2\frac{1}{54} - 81\times\frac{1}{486}\log_2\frac{1}{486} \\ &= \frac{1}{3}\log_2 3 + \frac{1}{3}\log_2 27 + \frac{1}{6}\log_2 54 + \frac{1}{6}\log_2 486, \end{aligned} \tag{8}$$

or

$$\boxed{\langle \text{\# of Qs} \rangle = \frac{1}{6}\log_2\left(9\times 729\times 54\times 486\right) \approx 4.56.} \tag{9}$$

Rounding Eq.(9) up to an integer number, we see that **it will take on average five questions** to determine the hidden number, presuming we implemented an optimal binary questioning strategy. Therefore we should certainly take the bet!

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2. **Personal Microstate and Macrostate Examples**

(a) (2 points) A fair coin is flipped 10 times.

- **Macrostate:** We can have a macrostate consisting of 4 heads and 6 tails.
- **Microstate:** One particular microstate of this macrostate could be the sequence of flips HTTHTTTHHT.

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(b) (2 points) We draw 10 balls from a bag filled with 40 distinguishable balls equally divided between red, orange, yellow, and green colors.

- **Macrostate:** We can have a macrostate consisting of 3 red balls, 5 orange balls, 1 yellow ball, and 1 green ball.
- **Microstate:** One particular microstate of this macrostate could be the balls $R_1,\ R_5,\ R_7,\ O_2,\ O_4,\ O_6,\ O_8,\ O_{10}, Y_9, G_7$.

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(c) (2 points) We roll three fair six-sided dice.

- **Macrostate:** We can have a macrostate consisting of dice whose values sum up to 11.
- **Microstate:** One particular microstate of this macrostate could be three dice each of which has one of the values 3, 6, 2

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3. **Gamma function (10 points)**

(a) We have the integral

$$\Gamma(N+1) = \int_0^\infty dx\, e^{-x}x^N. \tag{10}$$

Our goal is to relate this integral to the one defining $\Gamma(N)$. First, we implement integration by parts

$$\int_0^\infty dx\, f(x)\frac{d}{dx}g(x) = f(x)g(x)\Big|_0^\infty - \int_0^\infty dx\, g(x)\frac{d}{dx}f(x), \tag{11}$$

where we will take $f(x) = x^N$ and $dg(x)/dx = e^{-x}$. We then have $df(x)/dx = Nx^{N-1}$ and $g(x) = -e^{-x}$. The original Gamma function is then

$$\begin{aligned}\Gamma(N+1) &= \int_0^\infty dx\, e^{-x} x^N \\ &= -x^N e^{-x}\Big|_0^\infty + N \int_0^\infty dx\, e^{-x} x^{N-1} \\ &= N \int dx\, e^{-x} x^{N-1},\end{aligned} \tag{12}$$

where in the final equality we used the limit

$$\lim_{x\to\infty} x^N e^{-x} = 0. \tag{13}$$

By the definition of the gamma function, we therefore can conclude that

$$\boxed{\Gamma(N+1) = N\,\Gamma(N).} \tag{14}$$

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(b) Evaluating $\Gamma(N+1)$ for $N = 0$, we have

$$\boxed{\Gamma(1) = \int_0^\infty dx\, e^{-x} = -e^{-x}\Big|_0^\infty = 1.} \tag{15}$$

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(c) Using Eq.(14) iteratively and then using Eq.(15) for the final Gamma function, we have

$$\begin{aligned}\Gamma(N+1) &= N\Gamma(N) = N(N-1)\Gamma(N-1) \\ &= N(N-1)(N-2)\cdots 2\cdot 1\Gamma(1) \\ &= N(N-1)(N-2)\cdots 2\cdot 1.\end{aligned} \tag{16}$$

We therefore, find that $\Gamma(N+1)$ is simply N factorial: $\boxed{\Gamma(N+1) = N!.}$ Consequently, we can use the integral definition of $\Gamma(N+1)$ to define $N!$ for numbers N which are not exclusively integers.

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4. **Unusual Band**

(a) (5 points) We want to determine the entropy of a system with n_+ right-ward pointing links and n_- left-ward pointing links. We will use the Boltzmann definition of entropy in which, for Ω microstates in the system, the entropy is

$$S = k_B \ln \Omega. \tag{17}$$

Given that each link can point either to the right or to the left and there are N total links, the number of possible ways to have n_+ right-ward links is "N choose n_+". Thus the number of microstates is

$$\Omega = \binom{N}{n_+} = \frac{N!}{n_+!(N-n_+)!}. \tag{18}$$

Therefore, the entropy of this system is

$$\boxed{S(N,n_+) = k_B \ln \frac{N!}{n_+!(N-n_+)!}.} \tag{19}$$

For links, of which each is length ℓ, the total length of the rubber band is

$$L = \ell(n_+ - n_-) = \ell(2n_+ - N), \tag{20}$$

where we used $n_- = N - n_+$. Solving Eq.(20) for n_+ gives us

$$n_+ = \frac{1}{2}\left(N + L/\ell\right) = \frac{N}{2}\left(1 + L/\ell N\right). \tag{21}$$

Substituting Eq.(21) into Eq.(19), we find

$$\boxed{S = k_B \ln \frac{N!}{\left[\frac{N}{2}(1+L/\ell N)\right]!\left[\frac{N}{2}(1-L/\ell N)\right]!},} \tag{22}$$

which is the entropy written exclusively in terms of L and N.

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(b) (5 points) Stirling's approximation gives us

$$\ln N! = N\ln N - N + \frac{1}{2}\ln(2\pi N) + \mathcal{O}(N^{-1}). \tag{23}$$

Neglecting the $\frac{1}{2}\ln(2\pi N)$ term, we would like to use Eq.(23) to write Eq.(22) as an analytic function of L. First, we use a logarithmic identity to represent Eq.(22) as three terms:

$$S/k_B = \ln N! - \ln\left[\frac{N}{2}(1+L/\ell N)\right]! - \ln\left[\frac{N}{2}(1-L/\ell N)\right]! \tag{24}$$

Now, applying Eq.(23) to each term in Eq.(24), we have

$$\begin{aligned}
S/k_B &= N\ln N - N - \left[\frac{N}{2}(1+L/\ell N)\right]\ln\left[\frac{N}{2}(1+L/\ell N)\right] + \frac{N}{2}(1+L/\ell N) \\
&\quad - \left[\frac{N}{2}(1-L/\ell N)\right]\ln\left[\frac{N}{2}(1-L/\ell N)\right] + \frac{N}{2}(1-L/\ell N) + \mathcal{O}(\ln N) \\
&= N\ln N - \left[\frac{N}{2}(1+L/\ell N)\right]\ln\left[\frac{N}{2}(1+L/\ell N)\right] \\
&\quad - \left[\frac{N}{2}(1-L/\ell N)\right]\ln\left[\frac{N}{2}(1-L/\ell N)\right] + \mathcal{O}(\ln N) \\
&= N\ln N - \frac{N}{2}\left[\ln\frac{N}{2} + \ln\left(1+\frac{L}{\ell N}\right)\right] - \frac{L}{2\ell}\left[\ln\frac{N}{2} + \ln\left(1+\frac{L}{\ell N}\right)\right] \\
&\quad - \frac{N}{2}\left[\ln\frac{N}{2} + \ln\left(1-\frac{L}{\ell N}\right)\right] + \frac{L}{2\ell}\left[\ln\frac{N}{2} + \ln\left(1-\frac{L}{\ell N}\right)\right] + \mathcal{O}(\ln N) \\
&= N\ln N - \frac{N}{2}\ln\frac{N}{2} - \frac{N}{2}\ln\left(1+\frac{L}{\ell N}\right) - \frac{L}{2\ell}\ln\left(1+\frac{L}{\ell N}\right) \\
&\quad - \frac{N}{2}\ln\frac{N}{2} - \frac{N}{2}\ln\left(1-\frac{L}{\ell N}\right) + \frac{L}{2\ell}\ln\left(1-\frac{L}{\ell N}\right) + \mathcal{O}(\ln N)
\end{aligned} \tag{25}$$

In the second equality, we canceled the non-logarithm terms. In the third equality, we used the properties of logarithms to distribute the coefficients among multiple terms. In the final equality, we canceled the terms of the form $L/2\ell \ln(N/2)$. Further simplifying the final line, we obtain

$$\boxed{S/k_B = N\ln 2 - \frac{N}{2}\ln\left(1 - \frac{L^2}{\ell^2 N^2}\right) - \frac{L}{2\ell}\ln\frac{1+L/\ell N}{1-L/\ell N} + \mathcal{O}(\ln N),} \tag{26}$$

which is the desired result. ■.

(c) (5 points) For this part, we seek to compute the force given the definition

$$F = -T\frac{\partial S}{\partial L}. \tag{27}$$

From Eq.(26), we find

$$\begin{aligned} F &= -T\frac{\partial}{\partial L}k_B\left[N\ln 2 - \frac{N}{2}\ln\left(1 - \frac{L^2}{\ell^2 N^2}\right) - \frac{L}{2\ell}\ln\frac{1+L/\ell N}{1-L/\ell N} + \mathcal{O}(\ln N)\right] \\ &= -k_B T\left[-\frac{N}{2}\frac{-2L/\ell^2 N^2}{1-L^2/\ell^2 N^2} - \frac{1}{2\ell}\ln\frac{1+L/\ell N}{1-L/\ell N} - \frac{L}{2\ell}\left(\frac{1/\ell N}{1+L/\ell N} + \frac{1/\ell N}{1-L/\ell N}\right)\right] \\ &= -k_B T\left[\frac{L}{\ell^2 N}\frac{1}{1-L^2/\ell^2 N^2} - \frac{1}{2\ell}\ln\frac{1+L/\ell N}{1-L/\ell N} - \frac{L}{2\ell}\frac{2/\ell N}{1-L^2/\ell^2 N^2}\right] \\ &= -k_B T\left[\frac{L}{\ell^2 N}\frac{1}{1-L^2/\ell^2 N^2} - \frac{1}{2\ell}\ln\frac{1+L/\ell N}{1-L/\ell N} - \frac{L}{\ell^2 N}\frac{1}{1-L^2/\ell^2 N^2}\right], \end{aligned} \tag{28}$$

which leaves us with

$$\boxed{F = \frac{k_B T}{2\ell}\ln\frac{1+L/\ell N}{1-L/\ell N}.} \tag{29}$$

(d) (5 points) For this part of the problem, we seek to expand Eq.(29) to first order in L in the limit $L/N\ell \ll 1$. Using the Taylor series expansion of the logarithm,

$$\ln(1+x) = x + \mathcal{O}(x^2) \qquad \text{[For } |x| < 1\text{]}, \tag{30}$$

we have

$$\begin{aligned} F &= \frac{k_B T}{2\ell}\left[\ln(1+L/\ell N) - \ln(1-L/\ell N)\right] \\ &= \frac{k_B T}{2\ell}\left[L/N\ell + \mathcal{O}(L^2/\ell^2 N^2) + L/N\ell + \mathcal{O}(L^2/\ell^2 N^2)\right] \\ &= \frac{k_B T}{N\ell^2}L + \mathcal{O}(L^2/\ell^2 N^2). \end{aligned} \tag{31}$$

Therefore if $F \simeq KL$, then K is given by

$$\boxed{K = \frac{k_B T}{N\ell^2}.} \tag{32}$$

If we took F to be a constant in Eq.(31), and we were to heat up the system leading to a temperature increase, then L (the length of the rubber band) would decrease.

For the video (**Rubberband Thermodynamics**), we have a slightly different situation. In the video L is a constant, and the rubber band is heated up with a light bulb, leading to an increase in temperature. The end result is that F increases, and the mass is lifted slightly off the scale making it have a lower reading and a lower apparent weight.

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