Physics III – Workshop Problems – Introduction to Statistical Physics

On Taylor Series, Probability, and Combinatorics

Week Summary

- **Taylor Series:** For a function $f(x)$, if all the higher order derivatives of $f(x)$ exist, and we are considering a domain of x values for which the power series converges, then we can express $f(x)$ as

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \tag{1}$$

  where $k! = k(k-1)\cdots 2\cdot 1$. Eq.(1) is the "Taylor series expansion of $f(x)$ about the point $x = x_0$."

  Formally, determining whether Eq.(1) converges (and the exact domain of x values in which it converges) requires methods from calculus which will not be relevant for the course. We will only be expanding functions for which the domain of convergence is well known.

- **Random Variables:** A random variable is a quantity that can take on a discrete or a continuous range of values as possible outcomes of a random process or experiment. The outcome of a random experiment is called an **event**. Random variables can be discrete or continuous. **Discrete** random variables can only take on discrete spectrum of values (e.g., $1, 2, 3, 4, 5, 6$ for a die roll). **Continuous** random variables can take on a continuous spectrum of variables (e.g., from 0 sec to an infinite number of seconds for a dividing bacteria).

  - **Discrete random variables:** Discrete random variables can take on a discrete spectrum of values. We can denote an arbitrary value in this discrete spectrum as j. Each value is associated with a probability p_j to obtain that value from the experiment. These p_j satisfy the normalization requirement

$$1 = \sum_j p_j, \tag{2}$$

    where the sum is over all possible values of j. We can compute the average of a function $f(j)$ of the random variable by calculating the probability weighted sum of the function over all possible values of the random variable. Namely, denoting the average of $f(j)$ as $\langle f(j)\rangle$ we have

$$\langle f(j)\rangle = \sum_j f(j)\, p_j. \tag{3}$$

    Two important applications of Eq.(3) are to calculating the mean and the variance of a random variable:

$$\langle j\rangle = \sum_j j\, p_j \qquad \text{and} \qquad \sigma_j^2 = \langle j^2\rangle - \langle j\rangle^2. \tag{4}$$

  - **Continuous random variables:** Continuous random variables can take on a continuous spectrum of values. We can denote an arbitrary value in this continuous spectrum as x. Unlike in the discrete case, we cannot define the probability to be at a particular x, but we can define the probability density as a function of x. We label this probability density as $p(x)$ and define it as

    Probability density $p(x)$ (defined): If $p(x)$ is the *probability density* for a continuous random variable x, then, for Δx sufficiently small[1], the probability that we get a value of x

[1] In particular Δx is an infinitesimal quantity in the sense that $p(x)$ and $p(x + \Delta x)$ can be taken to represent the same value of the probability density.

between x_1 and $x_1 + \Delta x$ as an outcome for our random experiment is

$$\text{Probability that } x \text{ is between } x_1 \text{ and } x_1 + \Delta x = p(x_1)\Delta x. \qquad [\text{For } \Delta x \text{ sufficiently small }] \tag{5}$$

Eq.(5) only defines the probability to be within a certain interval if the probability density $p(x)$ can be taken to be constant within that interval. Generally, for intervals in which $p(x)$ varies, we have to integrate it from the starting point to the ending point of the interval in order to find the probability to be within that interval. For example, the probability that the outcome is in a domain between $x = A$ and $x = B$ is

$$\text{Probability that } x \text{ is between } A \text{ and } B = \int_A^B dx\, p(x). \tag{6}$$

For our continuous random variable, if the possible values it can take on are only in the domain between $x = x_0$ and $x = x_f$, then the probability of taking on any value in this domain must be 1. Namely, by Eq.(6), we have

$$1 = \int_{x_0}^{x_f} dx\, p(x). \tag{7}$$

Eq.(7) is the continuous analog of Eq.(2). With Eq.(6) and the defined range of values of the continuous random variable, we can define the probability that the random variable is less then or equal to X. We term this the **cumulative probability** and calculate it with

$$P(x \leq X) = \int_{x_0}^{X} dx\, p(x'). \tag{8}$$

Analogously to Eq.(3), we can define the average of a function of our continuous random variable:

$$\langle f(x) \rangle = \int_{x_0}^{x_f} dx\, f(x)p(x). \tag{9}$$

For example, the mean of the random variable is

$$\langle x \rangle = \int_{x_0}^{x_f} dx\, x\, p(x). \tag{10}$$

We can similarly define $\langle x^2 \rangle$, which together with Eq.(10), can be used to compute the variance of the random variable:

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2. \tag{11}$$

- **Permutations and Combinations:** If we have N distinct elements, the number of ways we *order* these elements into a list of length $k \leq N$ (that is, the number of *permutations* of length k), is

$$\frac{N!}{(N-k)!}, \tag{12}$$

where $N! = \prod_{i=1}^{N} i$. We note that $0! = 1$, so that for $k = N$, Eq.(12) is $N!$. The number of ways to *choose* amongst these elements to create a group with k elements (that is, the number of *combinations* of length k) is

$$\frac{N!}{k!(N-k)!} \tag{13}$$

We also denote Eq.(13) as $\binom{N}{k}$ or ${}_N C_r$, and we call it "N choose k".

1 Practice Problems

1. **Taylor Series**
   Using Eq.(1), compute the Taylor series for $\ln(1+x)$ about the point $x = 0$.

2. **Discrete Random Variables**
   We play a coin flip game in which we flip a coin and whenever we land on tails the game ends. Let's say the probability of landing on heads is p and the probability of landing on tails is $1-p$.

   (a) What is the probability that the game ends after k flips of the coin? Call this quantity probability $P(k;p)$

   (b) What is the minimum value of k possible? Is there a maximum value of k? Using these results, determine the normalization identity that $P(k;p)$ must satisfy.

   (c) What is $\langle k \rangle$ for this distribution? What is σ_k^2 for this distribution?

3. **Continuous Random Variables**
   We play a game of darts on a board which has a width from $x = -R$ to $x = +R$. The probability density associated with the x coordinate of the dart is

   $$p(x) = A(R^2 - x^2), \tag{14}$$

   where A is a constant which is determined by ensuring that $p(x)$ is properly normalized.

   (a) Plot $p(x)$ as a function of x. What must the area between $p(x)$ and the x axis be? What must A be? What is $\langle x \rangle$? What is σ_x^2?

4. **Number of Hands of Poker, Part I**
   For a deck with 52 cards the values 2, 3, 4, 5, 6, 7, 8, 9, 10, $\mathbb{J}$, $\mathbb{Q}$, $\mathbb{K}$, and $\mathbb{A}$ constitute the *rank* of a card. Whether the card is club (♣), diamonds (♦), hearts (♥), or spades (♠) constitutes the card's *suit*.

   Say we draw 5 cards from this 52 card deck. There are 52 ways we can choose the first card, 51 ways to choose the second card, 50 ways for the third, 49 ways for the fourth and 48 ways for the fifth. There are 5! ways to reorder the chosen cards such that we obtain the same hand. Therefore there are

   $$\frac{52 \times 51 \times 50 \times 49 \times 48}{5!} = 2,598,960 \tag{15}$$

   possible hands. Using similar reasoning we can count the possible hands in a game of poker.

   - **One pair:** A poker hand containing two cards of the same rank and three other cards of all different ranks. *Example:* $\mathbb{A}$♥ $\mathbb{A}$♠ 3♦ $\mathbb{J}$♥ 2♣. There are

     $$\frac{52 \times 3 \times 48 \times 44 \times 40}{2! \times 3!} = 1,098,240 \tag{16}$$

     possible one-pair hands.

   - **Three of a kind:** A poker hand containing three cards of the same rank and two cards of other ranks. *Example:* 4♥ 4♠ 4♦ 2♥ 9♣. There are

     $$\frac{52 \times 3 \times 2 \times 48 \times 44}{3! \times 2!} = 54,912 \tag{17}$$

     possible three-of-a-kind hands

   For the hand listed above, explain how the computed value correctly counts the number of possible hands.

2 Solutions

1. We will compute the Taylor series of $\ln(1+x)$ about the point $x=0$. To compute this Taylor series, we will apply the formula

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k. \tag{18}$$

For $f(x) = \ln(1+x)$ and $x_0 = 0$, Eq.(18) becomes

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{d^k}{dx^k} \ln(1+x)\Big|_{x=0}. \tag{19}$$

To finish the calculation of Eq.(19), we need to compute the kth derivative of $\ln(1+x)$, at $x=0$. The first thing to note is that the $k=0$ term (representing no derivatives) is zero because $\ln(1+0) = \ln 1 = 0$. Therefore, we can start the summation in Eq.(19) at $k=1$:

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{x^k}{k!} \frac{d^k}{dx^k} \ln(1+x)\Big|_{x=0}. \tag{20}$$

Now, we will seek a pattern in the derivatives. Computing the first, second, third, and fourth derivatives of $\ln(1+x)$, we have

$$\begin{aligned}
\frac{d}{dx}\ln(1+x) &= \frac{1}{1+x} \\
\frac{d^2}{dx^2}\ln(1+x) &= -\frac{1}{(1+x)^2} \\
\frac{d^3}{dx^3}\ln(1+x) &= -\frac{-2}{(1+x)^3} \\
\frac{d^4}{dx^4}\ln(1+x) &= -\frac{(-2)(-3)}{(1+x)^4}.
\end{aligned} \tag{21}$$

Recognizing the pattern and generalizing it, we can claim

$$\frac{d^k}{dx^k}\ln(1+x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}, \tag{22}$$

for $k \geq 1$. Setting $x=0$ in Eq.(22), and inserting the result into Eq.(20), gives us

$$\boxed{\ln(1+x) = \sum_{k=1}^{\infty} \frac{x^k}{k!}(-1)^{k-1}(k-1)! = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k},} \tag{23}$$

which is the Taylor series of $\ln(1+x)$ expanded about the point $x=0$. Although it is not clear from this derivation, Eq.(23) is valid for $|x| < 1$. To prove this result would require the ratio test from calculus[2].

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2. (a) We want to determine $P(k;p)$ the probability that our game ends after k coin flips. The game ends when we get our first tails, so if we flip the coin k times, it will only end after the k coin flips if we get $k-1$ heads and a final tails. Taking p to be the probability of getting heads and $1-p$ to

[2]Being precise, Eq.(23) is valid for $-1 < x \leq 1$.

be the probability of getting tails[3] we find that the probability of getting $k-1$ heads and 1 final tails is

$$\boxed{P(k;p) = p^{k-1}(1-p).} \tag{24}$$

(b) The minimum number of coin flips we can have before the game ends is $k = 1$. Theoretically, we can have an infinite number of coin flips before the game ends. In any case, we expect the number of heads to be somewhere between $k = 1$ and some infinite number. Therefore, we expect the sum of the probabilities of getting $k = 1, 2, \ldots$ and so on to infinity heads to be 1. That is,

$$1 = P(1;p) + P(2;p) + \cdots , \tag{25}$$

where the summation goes to infinity. Writing this using summation notation, we have

$$1 = \sum_{k=1}^{\infty} P(k;p) = \sum_{k=1}^{\infty} p^{k-1}(1-p). \tag{26}$$

We can check that the rightmost equation in Eq.(26) satisfies the normalization condition by using the geometric series identity:

$$\frac{1}{1-x} = \sum_{\ell=0}^{\infty} x \qquad \text{[For } |x| < 1.\text{]} \tag{27}$$

Considering Eq.(26), we can define $\ell = k - 1$, in the summation and factor our $1 - p$. Doing so, we have

$$(1-p)\sum_{\ell=0}^{\infty} p^{\ell} = (1-p)\frac{1}{1-p} = 1. \tag{28}$$

So Eq.(26) is indeed true.

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(c) To compute $\langle k \rangle$, we apply the standard formula for the mean of a random variable

$$\langle k \rangle = \sum_{k=1}^{\infty} P(k;p) = \sum_{k=1}^{\infty} kp^{k-1}(1-p) = (1-p)\sum_{k=1}^{\infty} kp^{k-1}. \tag{29}$$

To compute the final equality, we note that we have the geometric series identity

$$\frac{1}{1-p} = \sum_{k=0}^{\infty} p^k = \sum_{k=1}^{\infty} p^k + 1, \tag{30}$$

where in the second equality we isolated the $k = 1$ term to have a summation which starts at the same index as the lowest value of the geometric series. We can differentiate both sides of Eq.(30), to obtain

$$\begin{aligned} \frac{d}{dp}\frac{1}{1-p} &= \frac{d}{dp}\left(\sum_{k=1}^{\infty} p^k + 1\right) \\ \frac{1}{(1-p)^2} &= \sum_{k=1}^{\infty} kp^{k-1}. \end{aligned} \tag{31}$$

[3] It is p and $1-p$ because the probability of getting either heads or tails must be $p + 1 - p = 1$.

Returning to Eq.(29) and using Eq.(31) to compute the final equality, we have

$$\langle k\rangle = (1-p)\sum_{k=1}^{\infty} kp^{k-1} = (1-p)\frac{1}{(1-p)^2} = \boxed{\frac{1}{1-p}.} \tag{32}$$

Thus as $p \to 1$ (i.e., the probability of getting heads goes to 1), the mean number of coin flips we have before the game ends, goes to infinity. For example, for $p = 0.95$ representing a 95% probability of getting heads, the average number of coin flips is $\langle k\rangle = 1/.05 = 20$.

Now, to compute the variance of k we apply a similar procedure. To find this variance we need to compute $\langle k^2\rangle$ and insert it into the formula

$$\sigma_k^2 = \langle k^2\rangle - \langle k\rangle^2. \tag{33}$$

But instead of computing $\langle k^2\rangle$ directly, we will compute $\langle k(k-1)\rangle$ and use the result of this calculation to find $\langle k^2\rangle$. Doing so, we have

$$\begin{aligned}\langle k(k-1)\rangle &= \sum_{k=1}^{\infty} k(k-1)p^{k-1}(1-p)\\ &= p(1-p)\sum_{k=1}^{\infty} k(k-1)p^{k-2}.\end{aligned} \tag{34}$$

We factored $p(1-p)$ from the line above because doing so puts the equation in a form which is more useful to us. We can show that Eq.(34) is a more useful form for the equation by returning to the geometric series identity

$$\frac{1}{1-p} = \sum_{k=1}^{\infty} p^k + 1. \tag{35}$$

Differentiating both sides of this equation twice with respect to p, we have

$$\frac{2}{(1-p)^3} = \sum_{k=1}^{\infty} k(k-1)p^{k-2}. \tag{36}$$

Returning to Eq.(34), we then obtain for $\langle k(k-1)\rangle$

$$\langle k(k-1)\rangle = p(1-p)\frac{2}{(1-p)^3} = \frac{2p}{(1-p)^2}. \tag{37}$$

Because $k(k-1) = k^2 - k$, we ultimately have

$$\langle k^2\rangle - \langle k\rangle = \frac{2p}{(1-p)^2}, \tag{38}$$

or

$$\langle k^2\rangle = \frac{2p}{(1-p)^2} + \langle k\rangle = \frac{2p}{(1-p)^2} + \frac{1}{1-p} = \frac{1+p}{(1-p)^2}. \tag{39}$$

Finally, computing the variance gives us

$$\begin{aligned}\sigma_k^2 &= \langle k^2\rangle - \langle k\rangle^2\\ &= \frac{1+p}{(1-p)^2} - \frac{1}{(1-p)^2} = \boxed{\frac{p}{(1-p)^2}.}\end{aligned} \tag{40}$$

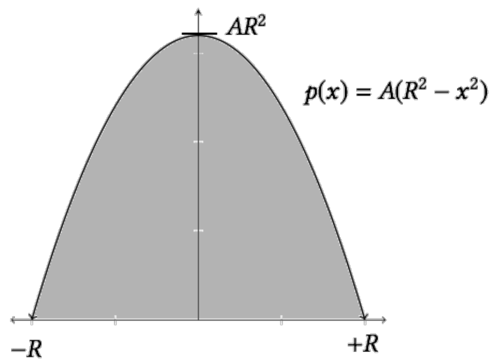

Figure 1: Plot of $p(x)$ in Problem 2

Considering Eq.(32) and Eq.(40) together, we can compute the relative error associated with repeated trials of this experiment. The ratio $\sigma_k/\langle k\rangle$ gives a measure of the size of the percentage deviation of the experimentally obtained number of flips from the theoretical value for number of flips we have before the game ends. From Eq.(40), we have for the standard deviation

$$\sigma_k = \frac{\sqrt{p}}{1-p}. \tag{41}$$

Therefore the ratio between the standard deviation and the mean is

$$\frac{\sigma_k}{\langle k\rangle} = \frac{\sqrt{p}}{1-p}\frac{1-p}{1} = \sqrt{p}. \tag{42}$$

Eq.(42) indicates that the relative error between theoretical mean and experimental mean decreases as $p \to 0$ and is at a maximum value of 1 when $p = 1$. This maximum value of 1 is an artifact of the results for variance and mean both of which become infinite as p goes to 1. This indicates that as the probability to get heads goes to 1, the average number of coin flips we expect to have until game-over becomes infinite. However, 1there is no reliable definition of standard deviation about an infinite mean.

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3. (a) A plot of $p(x)$, the probability density for the horizontal position of the dart, is shown in Fig. 1. The possible values of x range from $x = -R$ to $x = R$. Therefore, in order for $p(x)$ to be properly normalized (i.e., Eq.(7)), we require

$$1 = \int_{-R}^{R} dx\, p(x). \tag{43}$$

Eq.(43) indicates that the shaded area in Fig. 1 should be equal to 1. Computing Eq.(43), we have

$$\begin{aligned}\int_{-R}^{R} dx\, p(x) &= \int_{-R}^{R} dx\, A(R^2 - x^2) \\ &= A\int_{-R}^{R} dx\,(R^2 - x^2) \\ &= 2A\int_{0}^{R} dx\,(R^2 - x^2) \qquad \text{[Using the } x \to -x \text{ symmetry of the integrand.]}\end{aligned}$$

$$= 2A\left[R^2x - \frac{x^3}{3}\right]_0^R$$

$$= 2A\left(R^3 - \frac{R^3}{3}\right) = 4A\frac{R^3}{3}. \tag{44}$$

A is an undetermined constant whose value is fixed by the requirement that the total probability to find the dart at any x coordinate between $-R$ and R must be 1. Thus, by the normalization condition Eq.(43), we must have

$$\boxed{A = \frac{3}{4R^3}.} \tag{45}$$

From, here we can compute the mean. By definition (Eq.(10)), we have

$$\langle x \rangle = \frac{3}{4R^3}\int_{-R}^{R} dx\, x(R^2 - x^2). \tag{46}$$

However, if me make the change of variables $u = -x$, we find

$$\int_{-R}^{R} dx\, x(R^2 - x^2) = \int_{R}^{-R}(-du)\,(-u)(R^2 - u^2) = \int_{R}^{-R} du\, u(R^2 - u^2) = -\int_{-R}^{R} du\, u(R^2 - u^2). \tag{47}$$

Where, in the last equality, we used the fact that interchanging the limits of integration introduces a minus sign in front of the integral. Because the integration variable of a definite integral does not affect the final value of the integration, Eq.(47) presents an integral as equal to minus of itself. This can only be true if the integral is zero. Therefore,

$$\int_{-R}^{R} dx\, x(R^2 - x^2) = 0, \tag{48}$$

and $\boxed{\langle x \rangle = 0.}$

Since the mean is zero, the variance is simply $\sigma_x^2 = \langle x^2 \rangle$. Computing, this quantity (in a way analogous to how we confirmed the normalization of the probability density), we have

$$\langle x^2 \rangle = \int_{-R}^{R} dx\, x^2 A(R^2 - x^2)$$

$$= \frac{3}{4R^3}\int_{-R}^{R} dx\,(R^2x^2 - x^4)$$

$$= \frac{3}{2R^3}\int_{0}^{R} dx\,(R^2x^2 - x^4) \qquad \text{[Using the } x \to -x \text{ symmetry of the integrand.]}$$

$$= \frac{3}{2R^3}\left[R^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^R$$

$$= \frac{3}{2R^3}\left(\frac{R^5}{3} - \frac{R^5}{5}\right) = \frac{R^2}{5}. \tag{49}$$

Thus, since $\langle x \rangle = 0$, we have

$$\boxed{\sigma_x^2 = \frac{R^2}{5}.} \tag{50}$$

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4. We want to analyze and explain the various ways to obtain the stated hands

- **One pair:** A poker hand containing two cards of the same rank and three other cards of all different ranks. *Example:* A♥ A♠ 3♦ J♥ 2♣.

  We want to count the number of ways to select a hand with two cards of the same rank and three other cards of all different ranks. Out of a 52 card deck, we can choose any card to be the first card of our single pair. Thus there are 52 choices for the first card of a pair. For the second card of the pair, there are 3 other cards of the same rank, but different suit. Thus there are 52×3 possible choices for a *particular* ordering of one pair. However, the ordering of the two cards in the pair does not matter, so we divide this result by $2!$. Therefore, there are $52 \times 3/2!$ possible ways to create a pair from a 52 card deck.

  For the remaining three cards in the hand, we need each card to be a different rank from the rank of the original pair and from each other's rank. There are $52 - 4 = 48$ cards with ranks different from the ranks composing the pair. Thus there are 48 choices for the first of the three additional cards. Next, there are $52 - 4 - 4 = 44$ cards with ranks different from the ranks composing the pair and the first of the three additional cards. Thus there are 44 choices for the second of the three additional cards. By similar reasoning, there are 40 choices for the third of the three additional cards. Therefore there are $48 \times 40 \times 44$ ways to select a *particular* ordering of the three cards of different rank. However, the $3!$ ways to order these cards is not important. Therefore, to find the number of ways to create this three card set, where each card has a rank different from every other and different from the rank of the two pair, we divide $48 \times 40 \times 44$ by $3!$.

  Multiplying the number of ways to form the pair by the number of ways to form the remaining three cards, we find there are

  $$\frac{52 \times 3 \times 48 \times 44 \times 40}{2! \times 3!} = 1,098,240 \qquad (51)$$

  possible one-pair hands.

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- **Three of a kind:** A poker hand containing three cards of the same rank and two cards of other ranks. *Example:* 4♥ 4♠ 4♦ 2♥ 9♣.

  We want to find the number of ways to select three cards of the same rank and two cards of two other ranks. This problem is exactly analogous to the one pair problem except instead of forming a hand with only two cards of the same rank, we are forming a hand with only three cards of the same rank. In choosing the three cards of the same rank, any card of the 52 card deck can be the first card. Thus there are 52 possible choices for the first card. After this first card, there are 3×2 possible orderings for the next two cards of the same rank as the first card but of different suit. Thus, there are $52 \times 3 \times 2$ ways to create a *particular* ordering of a three of a kind from three cards. However, the $3!$ ways to order each particular ordering of the cards are equivalent at the level of the definition of the hand. So the number of ways to create any ordering of a three of a kind from three cards is $52 \times 3 \times 2/3!$.

  For the remaining two cards in the hand, there are $52 - 4 = 48$ choices for cards of rank different from the rank composing the three of a kind. Thus there are 48 choices for the first card in the remaining two cards. Similarly, there are $52 - 4 - 4 = 44$ choices for cards of rank different from the cards composing the three of a kind and the first card of the remaining two cards. Thus there are 44 choices for the second card. In all then, there are 48×44 ways to create a *particular* ordering of the remaining two cards. However, the ordering of these cards does not matter, so we divide this result by 2! to get $48 \times 44/2!$.

Multiplying the number of ways to create a three of a kind hand by the number of ways to choose the remaining two cards, we find there are

$$\frac{52 \times 3 \times 2 \times 48 \times 44}{3! \times 2!} = 54,912 \tag{52}$$

possible three-of-a-kind hands.

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