Review of Quantum Mechanics

1 Review

Postulates of Quantum Mechanics

1. **States:** At each instant the state of a physical system is represented by a ket $|\psi\rangle$ in the space of states.
2. **Observables:** Every observable attribute of a physical system is described by an operator that acts on the kets that describe the system.
3. **Values of Measurement:** The only possible result of the measurement of an observable A is one of the eigenvalues, a_n, of the corresponding operator $\hat{A}$.
4. **Probability of Measurement:** When a measurement of an observable A is made on a generic state $|\psi\rangle$, the probability of obtaining an **eigenvalue** a_n is given by the square of the inner product of $|\psi\rangle$ with the **eigenstate** $|a_n\rangle$, $|\langle a_n|\psi\rangle|^2$
5. **Collapse of State:** Immediately after the measurement of an observable A has yielded a value a_n, the state of the system is the normalized eigenstate $|a_n\rangle$.
6. **Time Evolution:** The form of the time evolution of a state is given by the **Schrödinger equation**:

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

   Where $\hat{H}$ is the energy/hamiltonian operator.

2 Spin

We will illustrate the postulates of quantum mechanics by considering the state defined by the spin (in the z direction) of an electron.

Aside: Spin was first discovered when physicists realized that the electron has a magnetic dipole moment which allows it to generate and interact with magnetic fields. Classically, a magnetic dipole can be generated by a rotating sphere with a certain charge density. Since the electron was known to have a certain charge and believed to be spherical, physicists made the assumption that it must also have some intrinsic angular momentum, spin, and this spin must allow it to act as a magnetic dipole. Further developments revealed that this classical intuition of the origin of spin is in fact incorrect and the "spinning" of the electron is very different from the "spinning" of objects we see in everday life.

1. **States:** The Spin State of a particle will be represented by the ket $|\chi_z\rangle$

$$|\chi_z\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \tag{1}$$

2. **Observables:** The Spin of the particle is described by the operator $\hat{S}_z$ which acts on $|\chi_z\rangle$.

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

3. **Values of Measurement:** The only possible result of the measurement of an observable $\hat{S}_z$ is one of the eigenvalues, $\hbar/2$ or $-\hbar/2$, of the corresponding operator $\hat{S}_z$.

$$\text{Measure the z-componentSpin of an Electron} \quad \Longrightarrow \quad \text{You can only get} + \frac{\hbar}{2} \text{ or} - \frac{\hbar}{2}$$

4. **Probability of Measurement:** When a measurement of an observable S_z is made on a generic state $|\chi_z\rangle$, the probability of obtaining an **eigenvalue** $+\hbar/2$ is given by the square of the inner product of $|\chi_z\rangle$ with the **eigenstate** $|+_z\rangle$, $|\langle +_z|\chi_z\rangle|^2$.

$$\begin{aligned} |+_z\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |\langle +_z|\chi_z\rangle|^2 &= \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 \\ &= |a|^2 \\ &= \text{Probability of measuring} + \hbar/2 \end{aligned}$$

5. **Collapse of State:** Immediately after the measurement of an observable S_z has yielded a value $+\hbar/2$, the state of the system is the normalized eigenstate $|+_z\rangle$.

6. **Time Evolution:** A spin particle existing in vacuum has no time evolution because nothing is interacting with it (i.e. $\hat{H} = 0$ so we must have

$$i\hbar\frac{d}{dt}|\chi_z\rangle = 0$$

and the state of the particle remains the same.

3 Problems

1. **Matrices:** Find the Eigenvalues and Normalized Eigenvectors of the following Matrices

   (a.) $\frac{1}{5}\begin{pmatrix} 3 & 4i \\ -4i & 3 \end{pmatrix}$ (b.) $\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & i & -1 \\ -i & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

2. **Eigenvalues and Eigenvectors of an Operator:** A two state system is characterized by the Hamiltonian operator

$$\hat{H} = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| - i|1\rangle\langle 2| + i|2\rangle\langle 1|\right)$$

   where a is a real number with the dimension of energy and $|1\rangle$ and $|2\rangle$ are orthonormal basis vectors. Find the energy eigenvalues and the corresponding normalized energy eigenkets (as linear combinations of $|1\rangle$ and $|1\rangle$).