June 28, 2010

MITES 2010

PHYSICS III

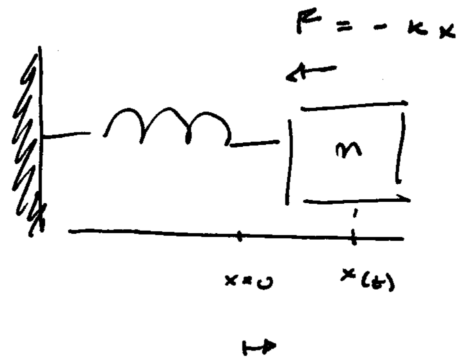
AGENDA

[1] RECAP

[2] LINEARITY & GEN SOLNS

[3] Complex solution & complex #s

[4] COUPLED OSCILLATIONS



[1] RECAP

WE WERE TRYING TO SOLVE THE SHM EQN

$$\ddot{x} + \omega^2 x = 0$$

WE GUESSED $x_1(t) = \sin(\alpha t)$ and FOUND

$$0 = \ddot{x}_1(t) + \omega^2 x_1(t) = -\alpha^2 \sin(\alpha t) + \omega^2 \sin(\alpha t)$$
$$= (-\alpha^2 + \omega^2) \sin(\alpha t)$$

SO $\alpha = \pm\omega$ & $\boxed{x_1(t) = \pm \sin(\alpha t)}$ ω

WE ALSO REALIZED WE COULD USE $x_2(t) = \cos(\alpha t)$ BECAUSE

$$\dot{x}_2(t) = -\alpha \sin(\alpha t)$$
$$\ddot{x}_2(t) = -\alpha^2 \cos(\alpha t)$$

$$\ddot{x}_2 + \omega^2 x_2 = -\alpha^2 \cos(\alpha t) + \omega^2 \cos(\alpha t)$$
$$= (-\alpha^2 + \omega^2) \cos(\alpha t)$$

SO $\alpha = \pm\omega$ $\boxed{x_2(t) = \cos(\omega t)}$

WE FOUND TWO SOLUTIONS. HOW DO WE FIND MORE? (1)

[2] LINEARITY & GEN. SOLNS

OUR TWO SOLNS ARE $x_1(t)$ & $x_2(t)$ WHICH SOLVE THE DIFFERENTIAL EQUATION AND THEREFORE SATISFY

$$\ddot{x}_1(t) + \omega^2 x_1(t) = 0$$

$$\ddot{x}_2(t) + \omega^2 x_2(t) = 0$$

HOW CAN WE GET OTHER SOLUTIONS?

1) <u>ADD</u>! LET $x_3(t) = x_1(t) + x_2(t)$

THEN $$\ddot{x}_3(t) + \omega^2 x_3(t) = \ddot{x}_1(t) + \ddot{x}_2(t) + \omega^2 (x_1(t) + x_2(t))$$

$$= \ddot{x}_1(t) + \omega^2 x_1(t) + \ddot{x}_2(t) + \omega^2 x_2(t)$$

$$= 0 + 0 = 0$$

SO $x_3(t)$ IS ANOTHER SOLUTION

WHAT ELSE CAN WE DO?

2) <u>MULTIPLY BY A CONSTANT</u>!

LET $x_4(t) = A x_1(t)$ $\quad x_5(t) = B x_2(t)$

THEN

$$\ddot{x}_4(t) + \omega^2 x_4(t) = A\ddot{x}_1 + \omega^2 A x_1$$ (Same as $x_4(t)$)

$$= A(\ddot{x}_1 + \omega^2 x_1)$$

$$= 0$$

SO WE CAN ADD TWO SOLUTIONS OR MULTIPLY THEM BY CONSTANTS TO OBTAIN OTHER SOLUTIONS.

THE MOST GENERAL SOLUTION IS A COMBINATION OF BOTH OPERATIONS

GENERAL SOLUTION

IF $x_1(t)$ & $x_2(t)$ are LINEARLY INDEPENDENT SOLUTIONS OF A D.E. THEN THE MOST GENERAL SOLUTION IS

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

OR

$$= \boxed{A x_1(t) + B x_2(t)}$$

WHERE A & B ARE ARBITRARY CONSTANTS ~~DEFIN~~ DEFINED BY A SPECIFIC SYSTEM

SO FOR SHO

$$x_1(t) = \cos(\omega t) \qquad x_2(t) = \sin(\omega t)$$

THE MOST GENERAL SOLUTION IS

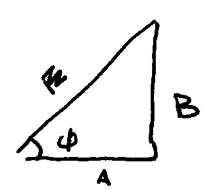
$$\boxed{x(t) = A\cos(\omega t) + B\sin(\omega t)}$$

HOW ELSE CAN WE WRITE THIS?

- SWITCH THE INDEPENDENT ~~CONSTAN~~ CONSTANTS A & B FOR TWO OTHER constants E & ϕ, WHERE

$$A = E\cos\phi$$

$$B = E\sin\phi$$



THEN

$$x(t) = E\cos\phi\cos(\omega t) + E\sin\phi\sin(\omega t)$$

$$= E\left(\cos\phi\cos(\omega t) + \sin\phi\sin(\omega t)\right)$$

FROM TRIG, WE KNOW $\cos(\alpha+\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

SO

$$\boxed{x(t) = E\cos(\omega t - \phi)}$$

NOTICE; WE STILL HAVE TWO INDEPENDENT CONSTANTS E & ϕ

THIS IS A GENERAL RESULT.

NTH ORDER DIFFERENTIAL EQUATION

$\downarrow$

N ARBITRARY & INDEPENDENT CONSTANTS

BUT!

THERE IS STILL ANOTHER WAY TO WRITE $x(t)$

[3] COMPLEX # & COMPLEX SOLUTION

INSTEAD OF A TRIG FUNCTION LET'S GUESS AN EXPONENTIA[L]

NAMELY

$$z(t) = e^{\alpha t}$$

☆ WE USE THE LABE[L] "z" BECAUSE $z(t)$ WILL TURN OUT TO BE A COMPLEX #.

PLUG THIS INTO OUR SHO EOM

$$\ddot{z}(t) + \omega^2 z(t) = \alpha^2 e^{\alpha t} + \omega^2 e^{\alpha t}$$

$$= (\alpha^2 + \omega^2) e^{\alpha t} = 0$$

SO $\alpha^2 = -\omega^2$

$$\alpha = \pm\sqrt{-\omega^2}$$

$$= \pm\sqrt{-1}\cdot\sqrt{\omega^2} = \pm i\omega \quad \text{WHERE } i = \sqrt{-1}$$

SO WE HAVE TWO SOLUTIONS $e^{+i\omega t}$, $e^{-i\omega t}$. IT TURNS OUT THAT THESE TWO ARE LINEARLY INDEPENDENT.

SO THE GENERAL SOLUTION IS

$$z(t) = Ce^{i\omega t} + De^{-i\omega t}$$

HOW IS THIS RESULT EQUIVALENT TO OUR FIRST?

ANS. THROUGH EULER'S FORMULA

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

HOW IS THIS TRUE?

- USUALLY PROVED WITH TAYLOR SERIES
- WE WILL TAKE IT AS FACT

☆ SUGGESTION: SET $f(\theta) = \cos\theta + i\sin\theta$. DIFFERENTIATE TO FIND $\frac{dF}{d\theta} = iF$. SOLVE DIFFERENTIAL EQUATION.

SO OUR $z(t)$ CAN BE WRITTEN AS.

$$z(t) = Ce^{i\omega t} + De^{-i\omega t}$$

$$= C(\cos(\omega t) + i\sin(\omega t)) + D(\cos(\omega t) - i\sin(\omega t))$$

$$= \underbrace{(C + D)}_{\text{complex \#}} \cos(\omega t) + i \underbrace{(C - D)}_{\text{complex \#}} \sin(\omega t)$$

$$z(t) = (A_1 + iA_2)\cos(\omega t) + (B_1 + iB_2)\sin(\omega t)$$

$$Re(z) = A_1\cos(\omega t) + B_1\sin(\omega t) = x(t)$$

CONCLUSION: $x(t)$ IS CONTAINED IN THE REAL COMPONENT OF $z(t)$. SIDENOTE: IT IS ALSO CONTAINED IN THE IMAGINARY COMPONENT

SO WE HAVE ~~FOUR~~ THREE WAYS OF WRITING THE SOLUTION TO

$$\ddot{x} + \omega^2 x = 0$$

$$\Downarrow$$

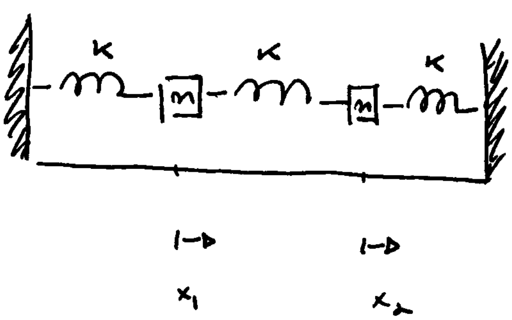
$$x(t) = C e^{i\omega t} + D e^{-i\omega t} \qquad \text{COMPLEX}$$

Real Part.

$$= A\cos(\omega t) + B\sin(\omega t) \qquad \text{REAL}$$

$$= R\cos(\omega t - \phi) \qquad \text{REAL}$$

[4] COUPLED OSCILLATIONS



TWO MASSES CONNECTED TO EACH OTHER WITH A SPRING OF CONSTANT k AND TO AN ADJACENT WALL WITH A SPRING OF CONSTANT k.

- WHAT IS THE RESULTING MOTION?

& REST LENGTH IS AT

$$x_1 = x_2 = 0$$

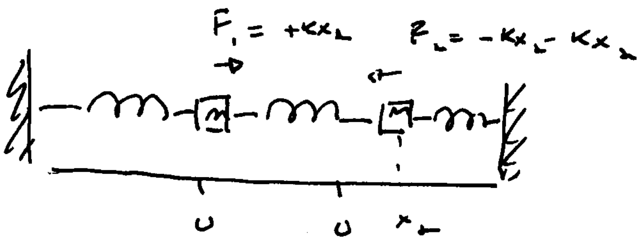
PROCEDURE

1) DERIVE EOMS.
2) SOLVE.

WE DERIVE THE EOM's FROM NEWTON's SECOND LAW AND A CONSIDERATION OF FORCES.

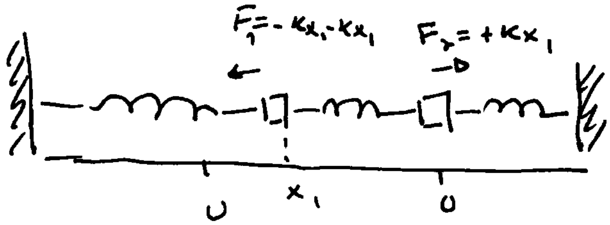
CONSIDERING THE MOTION OF BOTH MASSES IS DIFFICULT. LET'S CONSIDER ONE AT A ~~time~~ TIME

LET $x_1(t) = 0$; LET $x_2(t) > 0$



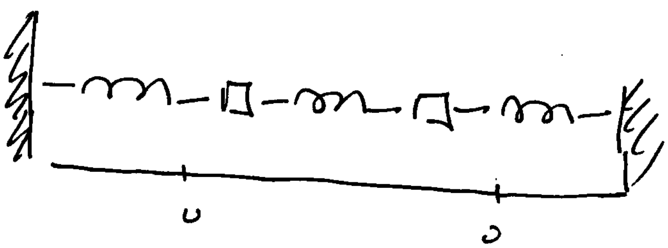
$$F_1 = +kx_2$$
$$F_2 = -2kx_2$$

LET $x_2(t) = 0$; LET $x_1(t) > 0$



$$F_1 = -2kx_1$$
$$F_2 = kx_1$$

GENERAL CASE : LET $x_1(t)$ & $x_2(t) \neq 0$ (SUM OF BOTH CASES)



$$F_1 = -2kx_1 + kx_2$$
$$F_2 = kx_1 - 2kx_2$$

NEWTON'S SECOND LAW

$$F_1 = m\ddot{x}_1 = -2kx_1 + kx_2$$
$$F_2 = m\ddot{x}_2 = kx_1 - 2kx_2$$

Let $\frac{k}{m} = \omega^2$

$$\boxed{\begin{aligned} \ddot{x}_1 &= -2\omega^2 x_1 + \omega^2 x_2 \\ \ddot{x}_2 &= \omega^2 x_1 - 2\omega^2 x_2 \end{aligned}}$$

EOMS

WE HAVE OUR EOMS, NOW WE "ONLY" NEED TO SOLVE FOR $x_1(t)$ & $x_2(t)$

WE WILL DO THIS IN TWO WAYS

1) MATRIX ~~METHOD~~ METHOD

2) LINEAR COMBINATION METHOD.

1) MATRIX METHOD

FROM MATRIX MULTIPLICATION $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \Leftrightarrow \begin{matrix} ax + by = e \\ cx + dy = f \end{matrix}$

SO

$$\ddot{x}_1 = -2\omega^2 x_1 + \omega^2 x_2$$

$$\ddot{x}_2 = \omega^2 x_1 - 2\omega^2 x_2$$

$$\Updownarrow$$

$$\boxed{\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$$

LIKE BEFORE, WE WILL GUESS AN EXPONENTIAL SOLUTION

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}$$

$$\frac{d^2}{dt^2}\Big\downarrow$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}$$

SO THE EOM NOW READS

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t} = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}$$

$$\alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} \cancel{e^{\alpha t}} = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \cancel{e^{\alpha t}}$$

EXPONENTIALS CANCEL & WE MULTIPLY THE LEFT SIDE BY THE IDENTITY MATRIX TO GET BOTH SIDES INTO THE "2×2 MATRIX $\cdot$ 2×1 MATRIX" FORMAT.

$$\alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

MOVE EVERYTHING TO ONE SIDE

$$0 = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} - \alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= \left[\begin{pmatrix} -2\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 \end{pmatrix} - \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} \right] \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} -2\omega^2-\alpha^2 & \omega^2 \\ \omega^2 & -2\omega^2-\alpha^2 \end{pmatrix}}_{\hat{M}} \underbrace{\begin{pmatrix} A \\ B \end{pmatrix}}_{\vec{a}}$$

$$= \hat{M}\vec{a}$$

SO $\boxed{\hat{M}\vec{a} = \vec{0}}$

THIS WAS A LONG PROCESS SO LET'S REMIND OURSELVES OF WHAT WE'RE LOOKING FOR.

WE GUESSED $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}$

WE WANT TO KNOW: A, B, & α

SO OUR EQUATION IS

$$\hat{M}\vec{a} = 0$$

HOW CAN WE SOLVE THIS? ANS: ASSUME WE KNOW $\hat{M}^{-1}$ WHERE $\hat{M}^{-1}$ is the inverse of $\hat{M}$ and THEREFORE SATISFIES

$$\hat{M}^{-1}\hat{M} = \mathbb{1} \leftarrow \text{Identity MATRIX}$$

THEN, WE MULTIPLY BOTH SIDES ~~BY ONE~~ ~~$\hat{M}^{-1}$~~ OF $\hat{M}\vec{a} = 0$ by $\hat{M}^{-1}$ AND WE FIND

$$\hat{M}^{-1}\hat{M}\vec{a} = \hat{M}^{-1}0$$

$$\mathbb{1}\vec{a} = 0$$

$$\vec{a} = 0 = \begin{pmatrix} A \\ B \end{pmatrix}$$

SO WE GET $A = B = 0$ AND NO MOTION EXISTS. THIS is CLEARLY NOT WHAT WE'RE LOOKING FOR.

SO WE <u>DO NOT</u> WANT $\hat{M}^{-1}$ TO EXIST.

WE CAN ENFORCE THIS BY MAKING $\det\hat{M} = 0$.

IT TURNS OUT

$$\hat{M}^{-1} = \frac{1}{\det\hat{M}}\left(\text{"SOME MATRIX"}\right)$$

SO IF $\det\hat{M} = 0$ THEN $\hat{M}^{-1}$ DOES NOT EXIST AND WE OBTAIN A NONZERO SOLUTION. SO WE WANT

$$\boxed{\det\hat{M} = 0}$$

WE HAVE A 2×2 MATRIX SO WE WILL NEED THE FORMULA FOR THE DETERMINANT OF A 2×2 MATRIX

WE HAVE THE $\pm$ CASE SO ONE SOLUTION IS FOUND TO BE

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(A_1 e^{+i\omega t} + A_2 e^{-i\omega t} \right)}$$

NOW FOR α_2

$$\underline{\alpha_2 = \pm i\sqrt{3}\,\omega}$$

$$\hat{M}_2 = \begin{pmatrix} -2\omega^2 - \alpha^2 & \omega^2 \\ \omega^2 & -2\omega^2 - \alpha^2 \end{pmatrix} = \begin{pmatrix} -2\omega^2 + 3\omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 + 3\omega^2 \end{pmatrix}$$

$$= \begin{pmatrix} \omega^2 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix}$$

SO

$$\hat{M}_2 \vec{a} \Rightarrow \begin{pmatrix} \omega^2 & \omega^2 \\ \omega^2 & \omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{matrix} \omega^2 A + \omega^2 B = 0 \\ \omega^2 A + \omega^2 B = 0 \end{matrix}$$

$$\boxed{A = -B}$$

SO WE FIND THAT OUR OTHER SOLUTION IS

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(B_1 e^{+\sqrt{3} i\omega t} + B_2 e^{-\sqrt{3} i\omega t} \right)}$$

USING OUR VARIOUS WAYS OF WRITING THE SOLUTION TO THE SHO EOM WE KNOW WE MAY WRITE

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_1 \cos(\omega t - \phi_1) \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} E_2 \cos(\sqrt{3}\omega t - \phi_2)$$

1F $\hat{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then $\boxed{\det \hat{O} = ad - bc}$

so

$$\hat{M} = \begin{pmatrix} -2\omega^2 - \alpha^2 & \omega^2 \\ \omega^2 & -2\omega^2 - \alpha^2 \end{pmatrix} \Rightarrow \det \hat{M} = (-2\omega^2 - \alpha^2)^2 - \omega^4 = 0$$

so

$$(-2\omega^2 - \alpha^2)^2 - \omega^4 = 0$$

$$-\omega^4 + 4\omega^4 + 4\omega^2\alpha^2 + \alpha^4 = 0$$

$$\alpha^4 + 3\omega^4 + 4\omega^2\alpha^2 = 0$$

$$(\alpha^2 + \omega^2)(\alpha^2 + 3\omega^2) = 0$$

$$\Downarrow$$

$$\boxed{\alpha_1 = \pm i\omega \qquad \alpha_2 = \pm\sqrt{3}\, i\omega}$$

THESE DEFINE THE CHARACTERISTIC FREQUENCIES OF THE MOTION.

WHAT DO α_1 & α_2 SAY ABOUT A & B?

LET'S CHOOSE α_1 FIRST

$\underline{\alpha_1 = \pm i\omega}$

$$\hat{M}_1 = \begin{pmatrix} -2\omega^2 - \alpha_1^2 & \omega^2 \\ \omega^2 & -2\omega^2 - \alpha_1^2 \end{pmatrix} = \begin{pmatrix} -2\omega^2 + \omega^2 & \omega^2 \\ \omega^2 & -2\omega^2 + \omega^2 \end{pmatrix}$$

$$= \begin{pmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{pmatrix}$$

SO WITH

$$\hat{M}_1 \vec{a} = 0 \Rightarrow \begin{pmatrix} -\omega^2 & \omega^2 \\ \omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$\begin{aligned} -\omega^2 A + \omega^2 B &= 0 \\ \omega^2 A - \omega^2 B &= 0 \end{aligned} \Rightarrow \boxed{A = B}$$

WHERE WHEN MOVING FROM COMPLEX EXPONENTIALS TO TRIG FUNCTIONS WE TOOK THE REAL PART OF THE EXPONENTIALS.

THESE TWO SOLUTIONS

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_1 \cos(\omega t - \phi_1)$$

&

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} E_2 \cos(\sqrt{3}\,\omega t - \phi)$$

ARE CALLED THE "NORMAL ~~COORDINATES~~ MODES" OF THE MOTION. THEY PHYSICALLY REPRESENT THE SIMPLEST NON-TRIVIAL MOTION OF THE SYSTEM. AND CONSEQUENTL EVERY MOTION CAN BE ~~BER~~ REPRESENTED AS A LINEAR COMBINATION OF THESE NORMAL MODES. THE MOST GENERAL SOLUTION IS A SUM OF THESE MODES

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(1)} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{(2)}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} E_1 \cos(\omega t - \phi_1) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} E_2 \cos(\sqrt{3}\,\omega t - \phi_2)$$

NOW FOR THE OTHER METHOD.

2) LINEAR COMBINATION

WE RETURN TO OUR EOMS

$$\ddot{x}_1 = -2\omega^2 x_1 + \omega^2 x_2$$

$$\ddot{x}_2 = \omega^2 x_1 - 2\omega^2 x_2$$

WE WILL FORM LINEAR COMBINATIONS (SUMS & DIFFERENCES) OF OUR TWO EQUATIONS, TO OBTAIN A SIMPLER RESULT.

1) <u>SUM</u>

$$\ddot{x}_1 + \ddot{x}_2 = -2\omega^2 x_1 + \omega^2 x_2 + \omega^2 x_1 - 2\omega^2 x_2$$

$$= -\omega^2 x_1 - \omega^2 x_2 = -\omega^2 (x_1 + x_2)$$

LET $u_+ = x_1 + x_2$

SO WE HAVE

$$\ddot{u}_+ = -\omega^2 u_+$$

WHICH IS THE SHO EOM. FROM OUR PREVIOUS ANALYSIS WE KNOW THAT THE RESULT HAS THE SOLUTION.

$$\boxed{u_+(t) = E_+ \cos(\omega t - \phi_+)}$$ WHERE E_+ & ϕ_+ ARE CONSTANTS

2) DIFFERENCE

$$\ddot{x}_1 - \ddot{x}_2 = -2\omega^2 (x_1 - x_2) + \omega^2 (x_2 - x_1)$$

$$= -2\omega^2 (x_1 - x_2) - \omega^2 (x_1 - x_2)$$

$$= -3\omega^2 (x_1 - x_2)$$

(14)

LET $u_- = x_1 - x_2$

SO WE HAVE

$$\ddot{u}_- = -3\omega^2 u_-$$

WITH THE SOLUTION

$$\boxed{u_-(t) = E_- \cos(\sqrt{3}\,\omega t - \phi_-)}$$

SO, WE FOUND u_+ & u_-, BUT WHAT ARE x_1 & x_2?

$$u_+ = x_1 + x_2$$

$$u_- = x_1 - x_2 \quad \Longrightarrow \quad \boxed{\begin{aligned} \frac{u_+ + u_-}{2} &= x_1 \\ \frac{u_+ - u_-}{2} &= x_2 \end{aligned}}$$

SO

$$x_1 = \tfrac{1}{2}u_+ + \tfrac{1}{2}u_-$$

$$= \frac{E_+}{2}\cos(\omega t - \phi_+) + \frac{E_-}{2}\cos(\sqrt{3}\,\omega t - \phi_-)$$

$$x_2 = \tfrac{1}{2}u_+ - \tfrac{1}{2}u_-$$

$$= \frac{E_+}{2}\cos(\omega t - \phi_+) - \frac{E_-}{2}\cos(\sqrt{3}\,\omega t - \phi_-)$$

WHICH CAN BE WRITTEN AS

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{E_+}{2}\cos(\omega t - \phi_+) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{E_-}{2}\cos(\sqrt{3}\,\omega t - \phi_-)$$

THIS RESULT IS EQUIVALENT TO THE ONE OBTAINED BY THE MATRIX METHOD.