MITE^{2}S 2010: Physics III
Survey of Modern Physics
Problem Set 0 Solutions

1 Problem 1. Relative velocity in one-dimension

(a)

Consider two trains that are moving towards each other on a single rail track(!). These two trains are set for a head-on collision. Train A is moving to the right at speed 100 m/s. Train B is moving to the left towards train A at speed 150 m/s.

(i).

For a person standing still inside train A, at what speed does train B seem to be moving towards him? This is called the speed of B *relative* to the person.

Solution: Physical intuition might tell us that we merely add the two speeds of the trains to obtain their relative velocity, but we can obtain the answer in a more rigorous way via the fundamental physical principle of **superposition** [1]. In superposition, we study our situation of interest by studying the system's individual component's and then summing each components effects at the end. We have a scenario in which both trains are moving towards one another with a nonzero speed. The motion of each train is independent of the other so we can consider each motion separately.

Case 1 (train A stationary; train B moving to the left with speed 150 m/s)
If train A is stationary and train B is moving to the left, towards train A, with a constant speed 150 m/s then, from the perspective of a person in train A, train B appears (and is) moving to the left at a speed $|v_B| = 150m/s$. Defining positive velocity as moving to the right, train B has a relative velocity of -150 m/s with respect to train A.

Case 2 (train A moving to the right with speed 100 m/s; train B stationary)
If train A is moving to the right, towards the stationary train B, with a speed of 100 m/s then, from the perspective of a person in train A, train B is the one moving towards train A with a speed of $|v_A| = 100m/s$. In case 1, we defined motion in which train B moved towards train A as motion with negative velocity so in case 2 it appears as though train B has a relative velocity of -100 m/s with respect to train A.

[1] Superposition is something you probably encountered in a study of the forces of Newtonian Mechanics. It is also found in Electrodynamics, Quantum Mechanics and generally anywhere the defining equations of the system are linear i.e. non quadratic

Superposition of Case 1 and Case 2 (train A moving to the right with speed 100 m/s; train B moving to the left with speed 150 m/s)
This is the scenario we have in the problem. Considering both situations studied in Case 1 and Case 2 we realize that the sum affect of each train's motion produces a relative velocity of

$$v_{rel} = \text{-150 m/s} + \text{-100 m/s} = \boxed{\text{-250 m/s}} \tag{1}$$

or, equivalently,

$$\boxed{\text{250 m/s to the left}}$$

(ii).

For a person walking to the front of train A (in the direction of the incoming train B) at a constant speed 2 m/s, at what speed does train B seem to be moving towards him? This is the speed of B *relative* to the person.

Solution: We use our principle of superposition again, but with an additional case to account for the motion of the person in train A. If the person in train A is moving towards train B with a speed of 2 m/s then, from the perspective of the person, train B is moving towards him with a speed of 2 m/s i.e. a velocity of -2 m/s. Including this effect with the result in (i) we obtain as our new relative velocity.

$$v_{rel} = \text{-2 m/s} + (\text{-250 m/s}) = \boxed{\text{-252 m/s}} \tag{2}$$

(iii).

For a person walking to the back of the train A (in the direction opposite of the incoming train B) at a constant speed 5 m/s, at what speed does train B seem to be moving towards him? This is called the speed of B *relative* to the person.

Solution: This situation is very similar to the one in (ii) except the velocity is in the opposite direction. As the person moves with a speed of 5 m/s to the left, train B appears to be moving with a speed 5 m/s away from the person i.e. to the right. So the additional relative velocity has the opposite sign.

$$v_{rel} = \text{5 m/s} + (\text{-250 m/s}) = \boxed{\text{-245 m/s}} \tag{3}$$

(b)

In one-dimension, we represent velocity by putting a sign in front of a number representing the object's speed. For example, v = 5 m/s represents an object moving to the right ($+x$

direction) at speed 5 m/s. u= -2 m/s represents an object moving to the left (-x direction) at speed 2 m/s. So when we say an object is moving in one-dimension (i.e. on a line) at velocity r, r can be either a positive or a negative number depending on the direction of the object's motion. Keeping this and your answers in (a) in mind, let's derive the general velocity addition rule for motion in one-dimension.

(i)

Consider two objects, A and B, that both move along a line. Let v_A and v_B be the velocities of objects A and B respectively. What is the velocity of B relative to A? Call this v_{BA}.

Solution: If we have two objects A and B separated in their x-coordinates by a distance x_{BA} then we may write the relation between their positions as

$$x_B = x_A + x_{BA}$$

Where we assume B is in front of A. Rearranging this result, to isolate x_{BA} on one side, we have the distance between the two objects

$$x_{BA} = x_B - x_A$$

Now, calculating the derivative with respect to time of the above equation gives us how the distance between the two objects change. In particular, it tells us how fast B is moving relative to A. Differentiation amounts to replacing the x coordinates with their respective velocities so we have.

$$\boxed{v_{BA} = v_B - v_A}$$

We can check this equation to make sure it matches our intuition. If $v_B = v_A$ then the two objects appear to be stationary relative to each other and we should get $v_{BA} = 0$, which we do. If $v_B > v_A$ then it would appear, relative to A, that B has positive velocity, which we also obtain from our formula. Lastly, if $v_B < v_A$ then we have the opposite case and we should get negative velocity, which is once again obtained from our result.

(ii)

Continuing the derivation from (iv), consider a third object C. It moves on a line parallel to A and B. Suppose it moves at a velocity v_{CB} relative to B. What is the velocity of C relative to A? Call this v_{CA}. This is called the Galilean law for velocity addition in one-dimension.

Solution: We use our above formula for the velocity of B relative to A $v_{BA} = v_B - v_A$, to find the formula for the velocity of C relative to B and the velocity of C relative to A
We find

$$\begin{aligned} v_{CB} &= v_C - v_B \\ v_{CA} &= v_C - v_A \end{aligned}$$

So writing v_{CA} in terms of v_{CB} and v_{BA} we have

$$\begin{aligned} v_{CA} &= v_C - v_B + v_B - v_A \\ &= \boxed{v_{CB} + v_{BA}} \end{aligned}$$

This formula can be understood by considering cantaloupes, apples, and bananas. Let all the fruits be thrown into the air. Assume the cantaloupe is moving relative to the banana with velocity v_{CB} and the banana is moving relative to the apple with velocity v_{BA} then the velocity of the cantaloupe relative to the apple is $v_{CB} + v_{BA} = v_{CA}$.

(c)

Let $x_A(t) = at^2 - bt + c$ and $x_B(t) = -dt^3 + bt^2$ be the position of trains A and B respectively at time t. They are both moving in one-dimension (along x-axis). What is the velocity of B relative to A (i.e. velocity of train B as seen by person standing still inside train A) as a function of time t?

Solution: We differentiate each $x(t)$ and apply the equation obtained in i.

$$\begin{aligned} \frac{d}{dt}x_A(t) &= 2at - b = v_A \\ \frac{d}{dt}x_B(t) &= -3dt^2 + 2bt = v_B \end{aligned}$$

and applying the relative velocity equation, we have

$$v_{BA} = v_B - v_A = \boxed{-3dt^2 + 2(b-a)t + b} \tag{4}$$

2 Problem 2. Motion in one-dimension and two-dimensions.

(a)

For the following anecdote, sketch a graph (by hand, without the aid of a calculator) of the object's position x, distance traveled d, velocity v, speed u, and acceleration a as a function of time t. Be sure to label the graph and include all the key features of the object's motion.

An ant managed to crawl up to an electrical transmission line and is constrained to move along the wire. At time t=0, the ant is at rest (sleeping because it is tired from crawling up the pole). It wakes up at time t=10 minutes, and begins to move forward at 2 cm/s. After moving forward at that constant speed for the next 2 minutes, it slows its walking speed at a rate of 0.1 cm/s^2 while still moving forward on the transmission wire. As soon as it reaches a forward moving speed of 0.5 cm/s, it abruptly brakes to a halt. It then rests for the next 20 minutes. Then, the ant starts to move backward at speed of 2 cm/s. It moves at that speed backwards until it has traveled a distance of 240 cm since it began moving backwards. At that point, it breaks to a halt at a rate of 0.5 cm/s^2.

Solution: On attached page.

(b)

Now suppose the ant were to fall vertically down to the ground after breaking to a halt (in (a)). If the telephone wire is height h above the ground, what is the ant's height y as a function of time t. (Acceleration due to gravity is g= 9.8 m/s^2; you can leave your answer in terms of the letter g). What is the ant's x-position as a function of time? What is the ant's acceleration in the x-direction as a function of time?

Solution: We know that the Force of gravity causes objects to fall. Specifically, from Newton's Second Law, we know that this force induces acceleration in the vertical direction.

$$\begin{aligned} ma_y(t) &= F_{y,net} \\ &= -mg \\ a_y(t) &= -g \end{aligned}$$

From the relations between acceleration, velocity, and position we know that our above result represents a **differential equation** which can be used to solve for $y(t)$. First, we begin with the defintion of acceleration in terms of the derivative of velocity.

$$\begin{aligned} a_y(t) = \frac{dv_y}{dt} &= -g \\ dv_y &= -g\,dt \\ \int_{v_0}^{v(t)} dv &= \int_0^t -g\,dt \\ v_y(t) - v_{0y} &= -gt \\ v_y(t) &= v_{0y} - gt \end{aligned}$$

And then we integrate once more to obtain the height as a function of time

$$\begin{aligned}
\frac{dy}{dt} &= v_y(t) \\
dy &= (v_{0y} - gt)\,dt \\
\int_{y_0}^{y(t)} dy &= \int_0^t (v_{0y} - gt)\,dt \\
y(t) - y_0 &= v_{0y}t - \frac{1}{2}gt^2 \\
y(t) &= y_0 + v_{0y}t - \frac{1}{2}gt^2
\end{aligned}$$

The last equation above represents the most general result for vertical motion in a gravitational field. To specialize this result to our situation we must make the arbitrary constants y_0 and v_{0y} conform to their expected values. We know that at the start of the motion (at t=0) the ant is at a height h and is at rest (it breaked to a halt). This information translates into the **initial conditions** $y(t=0) = h$, $v_{0y}(t=0) = 0$ which forces $y(t)$ above to take the form.

$$\boxed{y(t) = h - \frac{1}{2}gt^2}$$

To study the horizontal motion we use a similar analysis with the added constraint that there is no force in the horizontal direction. This fact therefore means that the acceleration in the x direction is zero, $\boxed{a_x = 0}$.

$$\begin{aligned}
a_x(t) = \frac{dv_x}{dt} &= 0 \\
dv_x &= 0\,dt \\
\int_{v_0}^{v(t)} dv &= 0 \\
v_x(t) - v_{0x} &= 0 \\
v_x(t) &= v_{0x}
\end{aligned}$$

And integrating to obtain $x(t)$

$$\begin{aligned}
\frac{dx}{dt} &= v_x(t) = v_{0x} \\
dx &= v_{0x}\,dt \\
\int_{x_0}^{x(t)} dx &= \int_0^t v_{0x}\,dt \\
x(t) - x_0 &= v_{0x}t \\
x(t) &= x_0 + v_{0x}t
\end{aligned}$$

Now, from part (a) we know that the ant is at a position $x = 14.75$ when it breaks to a halt. Also from this problem statement we know that the ant begins to fall from rest. This information translates into the inital conditions $x(t=0) = 14.75$ and $v_x(t=0) = 0$ which forces x to have the form

$$\boxed{x(t) = 14.75}$$

Physically, this equation tells us that as the ant falls, its horizontal position does not change.

(c)

Consider a new object - a baseball. A pitcher standing on the ground throws the ball at an angle $\theta = \pi/2$ radians with an initial speed of U_0. What is the total distance that the ball travels? How long does it take for it to come down to the ground? Plot the velocity of the ball v as a function of time t. Assume a constant acceleration due to gravity is g.

Solution: The ball is launched at an angle of $\pi/2$ or $90°$. This represents an exclusively vertical launch which makes this problem one-dimensional. We have already derived the equations which refer to this situation. Using the results from part (b) we have

$$a_y(t) = -g \qquad \Longrightarrow \qquad y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

and using the initial conditions $y(t=0) = 0$ $v_{y0}(t=0) = U_0$ our $y(t)$ function becomes

$$y(t) = U_0 t - \frac{1}{2}gt^2$$

The motion of the baseball governed by this equation is divided into two symmetric parts: an up part and a down part. The ball is thrown from the ground, rises to a maximum height, and then falls back to its initial height covering the same distance it traveled in its upwards flight. So to find the total distance the ball travels for its entire trajectory, we need only find the distance it travels in its upwards trajectory and multiply by two. Using the fact that the upwards trajectory of the ball ends when $v_y(t) = 0$ we can solve for the time that this occurs.

$$\begin{aligned} v_y(t_1) = 0 &= U_0 - gt_1 \\ t_1 &= \frac{U_0}{g} \end{aligned}$$

and the distance it has traveled up to time t_1 is

$$\begin{aligned} y(t_1) &= U_0 t_1 - \frac{1}{2}gt_1^2 \\ &= \frac{U_0^2}{g} - \frac{U_0^2}{2g} \\ &= \frac{U_0^2}{2g} \end{aligned}$$

To find the total distance the ball travels and the entire time it takes to complete its trajectory, we multiply $y(t_1)$ and t_1 by two because the upwards trajectory is the same as the downwards trajectory in terms of time and displacement. So we have

$$\boxed{\text{Total time} = \frac{2U_0}{g}}$$

$$\boxed{\text{Total Distance} = \frac{U_0^2}{g}}$$

Plot of velocity is on additional page.

(d)

Suppose now that the pitcher throws the ball at an angle $\theta = \pi/3$ with an initial speed u. Write down the time t dependent function representing the horizontal position $x(t)$ and the vertical position $y(t)$ of the ball. You can assume that $x(0) = y(0) = 0$. Also write down the horizontal and vertical velocities $v_x(t)$ and $v_y(t)$.

Solution: This problem was essential solved in our analysis of part (b). We need only specialize our previous equation to motion defined by an angle $\theta = \pi/3$. From the geometry of the right triangle which defines the components of a vector, we have the following equations

$$\begin{aligned} u &= \sqrt{v_{0x}^2 + v_{0y}^2} \\ u \sin\theta &= v_{0y} \\ u\cos\theta &= v_{0x} \end{aligned}$$

So our equation for motion in the y direction is

$$\begin{aligned} y(t) &= v_{0y}t - \frac{1}{2}gt^2 \\ &= u\sin\theta t - \frac{1}{2}gt^2 \\ &= \boxed{\frac{\sqrt{3}u}{2}t - \frac{1}{2}gt^2} \end{aligned}$$

and differentiation gives us $v_y(t)$

$$\frac{d}{dt}y(t) = \boxed{v_y(t) = \frac{\sqrt{3}u}{2} - gt}$$

Similarly for $x(t)$ we have

$$\begin{aligned} x(t) &= v_{0x}t \\ &= u\cos\theta t \\ &= \boxed{\frac{u}{2}t} \end{aligned}$$

and $v_x(t)$

$$\frac{d}{dt}x(t) = \boxed{v_x(t) = \frac{u}{2}}$$

(e)

At what angle θ should the pitcher throw the ball with speed u so that the ball travels the maximum possible horizontal distance before hitting the ground? Does this answer depend on the initial speed of the ball u? Again, you can assume that the acceleration due to gravity is g and is constant.

Solution: We need to find an angle θ which maximizes the horizontal displacement (let's call it D) of the baseball. First we need to find an explicit formula for D in terms of the parameters of the problem. Using the previously derived equations for two dimensional motion in a gravitational field, we have

$$y(t) = v_{0y}t - \frac{1}{2}gt^2 \tag{5}$$
$$x(t) = v_{0x}t \tag{6}$$

where

$$\begin{aligned} v_{0y} &= u\sin\theta \\ v_{0x} &= u\cos\theta \end{aligned}$$

The ball begins at $y = 0$ so when the ball hits the ground it is returning to this initial height. So, we need to solve for the time T it takes to return to this height and then plug this time into our $x(t)$ equation to find the total horizontal distance traveled. Using our $y(t)$ equation to solve for this time, we have

$$\begin{aligned} y(T) = 0 &= v_{0y}T - \frac{1}{2}gT^2 \\ &= v_{0y} - \frac{1}{2}gT \\ T &= \frac{2v_{0y}}{g} \end{aligned}$$

And plugging this time T into our $x(t)$ equation we have

$$\begin{aligned} D = x(T) &= v_{0x}T \\ &= \frac{2v_{0x}v_{0y}}{g} \\ &= \frac{2u^2}{g}\sin\theta\cos\theta \\ &= \frac{u^2}{g}\sin 2\theta \end{aligned}$$

Now, we need only realize that D is max when $\sin 2\theta = 1$. For motion which is constrained to exist between the angles 0 and $\pi/2$, $\sin 2\theta = 1$ when $2\theta = \pi/2$ or when $\boxed{\theta = \pi/4}$. Alternatively, we can use calculus to solve for the maximim of D. As a definition, we know that a function is maximum when the it has **zero slope** at a point with **negative curvature**. These two constraints are stated mathematically as

$$\frac{d}{dx}f(x) = 0$$
$$\frac{d^2}{dx^2}f(x) < 0$$

The second constraint is often forgotten but is necessary to separate local maxima from local minima, both of which have zero slope. So, computing these derivatives in the context of the problem, we find

$$\frac{d}{d\theta}D = \frac{2u^2}{g}\cos 2\theta = 0 \implies \theta = \pi/4$$

And checking the second condition

$$\frac{d^2}{d\theta^2}D(\theta = \pi/4) = -\frac{4u^2}{g}\sin(2\cdot\pi/4)$$
$$= -\frac{4u^2}{g} < 0$$

So both constraints are satisfied and $\theta = \pi/4$ is indeed a local maxima of D. This result means we should always throw a baseball with an angle of 45° in order to maximize the horizontal distance it travels.

(f)

Acceleration due to gravity is actually not a constant number. It changes as a function of height h above the ground. In particular, the acceleration due to gravity is $a(r) = -\frac{GM}{r^2}$, where r is the distance of the object from the Earth's center, G is a constant called 'Newton's gravitational constant', and M is the mass of the Earth. Why can we then assume that the acceleration due to gravity is the constant g? When is this approximation not valid? Give some quantitative reasoning. You can look up the numerical values of the radius of EarthR_E, M, and G to help you out.

Solution: The acceleration due to gravity at the earth's surface is a well known number. It's value written in terms of $a(r)$ is

$$a(R_E) = -\frac{GM}{R_E^2} = -g$$

where $g = 9.81m/s$, M is the mass of the earth, G is the gravitational constant, and R_E is the earth's radius. When we are close to the earth's surface we may consider our radial position r to be approximately R_E. The point of this problem is to quantitatively consider the domain of validity of this approximation. To consider distances which extend farther than the surface of the earth we add a small amount h to earth's radius R_E and define $r = R_E + h$. We then invoke the **linear approximation**[2]. In the linear approximation we use value of a function and its derivative at a certain point to find the value of the function at other points. For example, a function $f(x)$ and its derivative $f'(x)$ are known at the point x_0. To approximate the function at other points x, which are near x_0, we use

$$f(x) = f(x_0) + f'(x - x_0)(x - x_0)$$

This formula comes from rearranging the definition of the approximate derivative

$$\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0)$$

[2]This can also be considered as a Taylor Series Approximation in which we keep only the first order term.

So, using this result, we may approximate $a(r)$ for distances $r = R_E + h$ near R_E

$$\begin{aligned} a(R_E + h) &\approx a(R_E) + a'(R_E)h \\ &= -\frac{GM}{R_E^2} + 2\frac{GM}{R_E^3}h \\ &= -\frac{GM}{R_E^2}\left(1 - \frac{2h}{R_E}\right) \\ &= \boxed{-g\left(1 - \frac{2h}{R_E}\right)} \end{aligned}$$

Using the value of $R_E = 6.37 \times 10^6 m$ we can understand why $-g$ is such a good approximation for everyday heights. The Empire State Building (including the antenna) is roughly 450 m. Using this value as our height h and plugging this into our result, we have

$$\begin{aligned} a(R_E + 450\text{m}) &\approx -g\left(1 - \frac{2 \cdot 450m}{R_E}\right) \\ &= -g\left(1 - 1.41 \times 10^{-4}\right) \\ &= -g(0.999859) \\ &= -9.808 \end{aligned}$$

which still doesn't represent a marked deviation from our understood -9.81 result; even at the very top of one of America's tallest buildings g is still g. Typically, there is a "5% rule" in the sciences which states that experimental values are allowed to deviate within 5 % of their accepted values. So allowing for this deviation we have

$$\begin{aligned} .95 &\leq 1 - \frac{2h}{R_E} \\ \frac{2h}{R_E} &\leq 0.05 \\ h &\leq .025 \times R_E \\ h &\leq 1.59 \times 10^5 m \end{aligned}$$

So as long as we remain a height $h < 1.59 \times 10^5$ then we have an experimentally insignificant deviation from our accostomed value of g. As a measure of reference, this height is about as tall as 300 Empire State buildings stacked, head to toe, on top of each other.