MITE^{2}S 2010: Physics III
Survey of Modern Physics
Problem Set 1 Solutions

Exercises

1. **Exercise 1: Most elementary particles are unstable: they disintegrate after a characteristic lifetime that varies from one species to the next (e.g., lifetime of a neutron is 15 min, of a muon, $2 \times 10^{-6} s$). These are the lifetimes of particles at rest. When particles are moving at speeds close to the speed of light c, they last much longer, for their internal clocks (whatever it is that tells them when their time is up) are running slow, in accordance with time-dilation. As an example, consider a muon that is traveling through the laboratory at three-fifths the speed of light. How long does it last?**

   Solution: The lifetime of the particle as measured when the particle is at rest defines how long the particle can exist. When the particle begins to move we must take into account relativity and the time that the particle experiences in its own frame becomes shorter than the time we observe on earth. In particular, denoting Δt_{MUON} as the time that the muon experiences and Δt_{EARTH} as the time we measure on earth, we find.

$$\begin{aligned}
\Delta t_{EARTH} &= \gamma \Delta t_{MUON} \\
&= \frac{1}{\sqrt{1 - v^2/c^2}} \Delta t_{MUON} \\
&= \frac{1}{\sqrt{1 - 9/25}} \Delta t_{MUON} \\
&= \frac{\Delta t_{MUON}}{\sqrt{16/25}} \\
&= \frac{5}{4}(2 \times 10^{-6} s) \\
&= \boxed{2.5 \times 10^{-6} s}
\end{aligned}$$

   So, according to the observers on earth, the muon has a longer lifetime than its expected lifetime.

2. **Exercise 2: A rocket ship leaves the earth at a speed of $\frac{3}{5}c$. When a clock on the rocket says 1 hour has elapsed, the rocket ship sends a light signal back to earth.**
   (a) According to earth clocks, when was the signal sent?
   (b) According to earth clocks, how long after the rocket left did the signal arrive back on earth?
   (c) According to the rocket observer, how long after the rocket left did the signal arrive back on earth?

   Solution: (a) The rocket acts as a reference frame moving with velocity v_R with respect to our stationary reference frame on earth. Then, according to time dilation, time must pass more quickly on earth than on the rocket by a factor of γ. Thus, labeling Δt_E as the time as measured by earth for the rocket to send the signal and Δt_R as the corresponding time for the rocket, we have

$$\begin{aligned}\Delta t_E &= \gamma \Delta t_R \\ &= \frac{\Delta t_R}{\sqrt{1-(v_R/c)^2}} \\ &= \frac{\Delta t_R}{\sqrt{1-9/25}} \\ &= \frac{\Delta t_R}{\sqrt{16/25}} \\ &= \frac{5}{4}(1\text{ hr}) \\ &= 1.25\text{ hrs}\end{aligned}$$

So in earth's frame, the signal was sent $\boxed{1.25\text{ hrs}}$ after the rocket left.

(b) To find the total time it takes the signal to arrive on earth after the rocket is launched, we must add the time that it takes the rocket to reach the point where it transmits the signal and the time that it takes the signal to get to earth. We calculated the first time, Δt_E, in (a), so we need only use this result to find the total distance of travel and the corresponding time it would take a light signal to reach earth. Call the time for the light signal to travel to earth $\Delta t_{L,E}$, then we must have

$$\begin{aligned}c\Delta t_{L,E} &= \text{Total Distance} \\ &= v_R \Delta t_E \\ &= \frac{3}{5}c \cdot \frac{5}{4}\text{hrs} \\ &= \frac{3}{4}(c \cdot \text{hrs}) \qquad \Longrightarrow \qquad \Delta t_{L,E} = \frac{3}{4}\text{ hrs}\end{aligned}$$

Therefore the total time that it takes the light signal to reach earth is

$$\Delta t_{\text{Total},E} = \Delta t_{L,E} + \Delta t_E = \frac{3}{4}\text{hrs} + \frac{5}{4}\text{hrs} = \boxed{2\,\text{hrs}}$$

(c) The time it takes the signal to reach earth in *earth's frame* is $\Delta t_{\text{Total},E} = 2$ hrs. With this result, we can use the time dilation formula to find the corresponding time in the rocket's frame. However for (c), we are considering the problem from the *rocket's frame* in which the rocket is stationary and earth is moving away from the rocket with speed v_R. In this case the correct time dilation formula is $\Delta t_{\text{Total},R} = \gamma \Delta t_{\text{Total},E}$ in which time appears to run slow on earth relative to the rocket's time. So, for the total time in the rocket's frame, we have

$$\begin{aligned}\Delta t_{\text{Total},R} &= \gamma \Delta t_{\text{Total},E} \\ &= \frac{5}{4} \cdot 2\text{ hrs} \\ &= \boxed{2.5\,\text{hrs}}\end{aligned}$$

We can also obtain this result by *only* considering the problem from rocket's perspective. In its own frame, $\Delta t_R = 1$ hr passes before the rocket sends the signal to earth. At this time, the earth is a distance $v_R \Delta t_R = \frac{3}{5}(c \cdot \text{hrs})$ away from the rocket. As the signal travels to earth, earth continues to move away from the rocket from the rocket's perspective. So the light signal must travel an extra distance $v_R \Delta t_{L,R}$ in order to reach earth, where $t_{L,R}$ is the time it takes light to reach earth from the rocket's perspective. So, in order to find this time $\Delta t_{L,R}$ we must solve

$$\begin{aligned} v_R \Delta t_{L,R} + \frac{3}{5}(c \cdot \text{hrs}) &= c \Delta t_{L,R} \\ \frac{3}{5}(c \cdot \text{hrs}) &= (c - v_R) \Delta t_{L,R} \\ \Delta t_{L,R} &= \frac{\frac{3}{5}(c \cdot \text{hrs})}{c - v_R} \\ &= \frac{\frac{3}{5}(c \cdot \text{hrs})}{c - 3/5c} \\ &= \frac{3/5}{2/5}\,\text{hrs} = \frac{3}{2}\,\text{hrs} \end{aligned}$$

So, from the rocket's perspective, the total time it takes the signal to reach earth is

$$\Delta t_{\text{Total},R} = \Delta t_{L,R} + \Delta t_R = \frac{3}{2}\,\text{hrs} + 1\text{hr} = \boxed{2.5\,\text{hrs}}$$

Which is the same result we obtained by time dilation.

3. **Exercise 3: A Lincoln Continental is twice as long as a VW Beetle, when they are at rest. As the Continental overtakes the VW, going through a speed trap, a (stationary) policeman observes that they both have the same length that the Lincoln Continental is twice as long as the . The VW is going at half the speed of light. How fast is the Lincoln going? Leave your answer as a multiple of c.**

   Solution: We know that the Lincoln Continental is twice as long as the VW Beetle when both are at rest ($L_{LC} = 2L_{VW}$). We also know that from the ground frame they have the same length ($\bar{L}_{LC} = \bar{L}_{VW}$) when they are moving with speeds v_{LC} and v_{VW} respectively. Lastly, we know that $v_{VW} = c/2$. Using this information, we can solve for the velocity of the lincoln continenental v_{LC}.

$$\begin{aligned} \frac{L_{LC}}{\gamma_{LC}} &= \frac{L_{VW}}{\gamma_{VW}} \\ \frac{2L_{VW}}{\gamma_{LC}} &= \frac{L_{VW}}{\gamma_{VW}} \\ \frac{2}{\gamma_{LC}} &= \frac{1}{\gamma_{VW}} \\ 2\sqrt{1 - \frac{v_{LC}^2}{c^2}} &= \sqrt{1 - \frac{v_{VW}^2}{c^2}} \\ 4\left(1 - \frac{v_{LC}^2}{c^2}\right) &= \left(1 - \frac{1}{4}\right) \\ \left(1 - \frac{v_{LC}^2}{c^2}\right) &= \frac{3}{16} \\ \frac{v_{LC}^2}{c^2} &= 1 - \frac{3}{16} \\ &= \frac{13}{16} \quad \Longrightarrow \quad \boxed{v_{LC} = \frac{\sqrt{13}}{4}c} \end{aligned}$$

4. **Exercise 4: A sailboat is manufactured so that the mast leans at an angle $\bar{\theta}$ with respect to the deck. An observer standing on a dock sees the boat go by at speed v. What angle does this observer say the mast makes?**

   Solution: A fundamental effect of special relativity is that lengths (of a moving object) parallel to the direction of motion are contracted. So, in this problem we realize that the horizontal extent of the bottom ray of the angle must be contracted due to the motion of the boat. Labeling the length of the mast as L we can write the height $\bar{H}$ and original horizontal extent $\bar{X}$ as

$$\begin{aligned}\bar{H} &= L\sin\bar{\theta} \\ \bar{X} &= L\cos\bar{\theta}\end{aligned}$$

   Due to length contraction, the value of $\bar{X}$ becomes the smaller value X in the moving reference frame and we find that the relation between X and $\bar{X}$ is

$$X = \frac{\bar{X}}{\gamma}$$

   So that $X = L\cos\bar{\theta}/\gamma$. Lengths perpendicular to the direction of motion are not changed by moving to different reference frames so we find that $H = \bar{H} = L\sin\bar{\theta}$. Using this information we can define the angle θ of the mast in the moving reference frame as

$$\begin{aligned}\tan\theta &= \frac{H}{X} \\ &= \frac{L\sin\bar{\theta}}{L\cos\bar{\theta}/\gamma} \\ &= \gamma\tan\bar{\theta} \quad\Longrightarrow\quad \boxed{\theta = \tan^{-1}(\gamma\tan\bar{\theta})}\end{aligned}$$

   When we get an algebraic result such as this one, it is useful to check **limiting cases** to make sure that our formula correctly predicts what physical intuition tells us should be true. For example, we will consider the formula for $v = 0$ and $v \to c$ and make sure it provides a result which mirrors what we think should happen. For $v = 0$ we know that the sailboat is not moving and therefore length contraction is absent. Therefore θ should be equal to $\bar{\theta}$, which is what we find when we plug $v = 0 \Leftrightarrow \gamma = 1$ into our formula. Conversely, as $v \to c$ the horizontal length is further contracted until it only appears as though we have a stick of length L standing straight up. This reasoning matches the $\theta = \pi/2$ result we get if we plug $v = c \Leftrightarrow \gamma = \infty$ into our above formula.

5. **Exercise 5: In class, we derived how the velocities in the x direction transform when you go from a moving frame $\bar{O}$ to a rest frame O. Derive the analogous formulas for the velocities in the y and z directions.**

   Solution: In order to derive the velocity transformation law for the y and z velocities, we must first collect the relevant Lorentz transformations for transforming between O and $\bar{O}$.

$$\begin{aligned}\Delta y &= \Delta\bar{y} \\ \Delta z &= \Delta\bar{z} \\ \Delta t &= \gamma\left(\Delta\bar{t} + \frac{v_x\Delta\bar{x}}{c^2}\right)\end{aligned}$$

Computing the velocity in the y direction for the O frame, we find

$$\begin{aligned}
u_y &= \lim_{\Delta\bar{t}\to 0} \frac{\Delta y}{\Delta t} \\
&= \lim_{\Delta\bar{t}\to 0} \frac{\Delta\bar{y}}{\gamma(\Delta\bar{t} + v_x\Delta\bar{x}/c^2)} \\
&= \lim_{\Delta\bar{t}\to 0} \frac{\Delta\bar{y}/\Delta\bar{t}}{\gamma(\Delta\bar{t}/\Delta\bar{t} + v_x\Delta\bar{x}/\Delta\bar{t}/c^2)} \\
&= \boxed{\frac{\bar{u}_y}{\gamma(1 + v_x\bar{u}_x/c^2)}}
\end{aligned}$$

where we divided the numerator and denominator by $\Delta\bar{t}$ in the third line.
The relative velocity between the two frames O and $\bar{O}$ exists only in the x direction, so all directions (i.e. the y and z directions) which are perpendicular to the motion between the frames do not undergo transformations as we move from one frame to another. So the derivation of u_z will be the same as the derivation for u_y and we have

$$\boxed{u_z = \frac{\bar{u}_z}{\gamma(1 + v_x\bar{u}_x/c^2)}}$$

6. **Exercise 6: A and B travel at $\frac{4}{5}c$ and $\frac{3}{5}c$ with respect to the ground. How fast should c travel so that she sees A and B approaching her at the same speed? What is this speed?**

   Solution: Let us label the velocity of A relative to C as $+u$. Then, according to the problem, the velocity of B relative to A must be $-u$. Using the relativistic transformations for velocity we then have

$$\begin{aligned}
+u &= \frac{v_C - v_A}{1 - \frac{v_C v_A}{c^2}} \\
-u &= \frac{v_C - v_B}{1 - \frac{v_B v_A}{c^2}}
\end{aligned}$$

   Where the letters label the velocities of the corresponding particle. From the problem statement we know $v_B = 3c/5$ and $v_A = 4c/5$ and we are trying to find v_C so, we have

$$\begin{aligned}
u &= u \\
\frac{v_C - v_A}{1 - \frac{v_C v_A}{c^2}} &= \frac{v_B - v_C}{1 - \frac{v_B v_A}{c^2}} \\
\frac{v_C - \frac{4c}{5}}{1 - \frac{4v_C}{5c}} &= \frac{\frac{3c}{5} - v_C}{1 - \frac{3v_C}{5c}} \\
\left(v_C - \frac{4c}{5}\right)\left(1 - \frac{3v_C}{5c}\right) &= \left(1 - \frac{4v_C}{5c}\right)\left(\frac{3c}{5} - v_C\right) \\
v_C - \frac{4c}{5} + \frac{12v_C}{25} - \frac{3v_C^2}{5c} &= -v_C + \frac{4v_C^2}{5c} - \frac{12v_C}{25} + \frac{3c}{5} \\
0 &= -2v_C + \frac{7v_C^2}{5c} - \frac{24v_C}{25} + \frac{7c}{5} \\
&= \frac{7v_C^2}{5c} - \frac{74v_C}{25} + \frac{7c}{5} \\
&= 1/25c\left(35v_C^2 - 74v_C c + 35c^2\right) \\
&= (5v_C - 7c)(7v_C - 5c)
\end{aligned}$$

The last line produces two solutions: $v_C = 7c/5$ and $v_C = 5c/7$. But, since the speed of an object can never exceed the speed of light the first solution is extraneous and we have $\boxed{v_C = 5c/7}$

7. **Exercise 7: A train with a rest length L moves at speed $5c/13$ with respect to the ground. A ball is thrown from the back of the train to the front. The speed of the ball with respect to the train is $c/3$. As viewed by someone on the ground, how much time does the ball spend in the air, and how far does it travel?**

   Solution: In order to find the time and distance of travel of the ball we must first find the speed of the ball relative to the ground. So, we use the relativistic velocity addition formula:

$$\begin{aligned} v_B &= \frac{\bar{v}_B + v_T}{1 + \frac{\bar{v}_B v_T}{c^2}} \\ &= \frac{c/3 + 5c/13}{1 + \frac{5\cdot 1}{13\cdot 3}} \\ &= c\frac{28/39}{1 + \frac{5}{39}} \\ &= c\frac{28/39}{44/39} \\ &= c\frac{7}{11} \end{aligned}$$

   Now, from the ground frame we know that as the ball is thrown, the train continues to move so that the ball has to travel an extra distance to reach the other side. We also know that due to special relativity, the length of the train is contracted in our frame. So, denoting t as the time it takes the ball to reach the other side of the train, we find

$$\begin{aligned} v_B t &= \frac{L}{\gamma} + v_T t \\ (v_B - v_T)t &= \frac{L}{\gamma} \\ t &= \frac{L}{\gamma(v_B - v_T)} \\ &= \frac{L}{c(7/11 - 5/13)}\sqrt{1 - \frac{v_T^2}{c^2}} \\ &= \frac{L}{c(36/143)}\sqrt{1 - 25/169} \\ &= \frac{143L}{36c}\sqrt{144/169} \\ &= \boxed{\frac{11L}{3c}} \end{aligned}$$

   And therefore the total distance the ball travels is $\boxed{v_B t = \frac{7}{3}L}$.

8. **Exercise 8: In a given reference frame, event 1 happens at $x = 0$, $ct = 0$, and event 2 happens at $x = 2$, $ct = 1$. Find a frame in which the two events are simultaneous.**

   Solution: One of the fundamental effects of relativity is the loss of simultaneity. A corollary to this effect, potentially called the restoration of simultaneity, is depicted in this problem. Our job is to find another frame $\bar{O}$ in which the two events $(x_1, t_1) = (0, 0)$ and $(x_2, t_2) = (2, 1/c)$ occur simultaneously (i.e. $\bar{t}_1 = \bar{t}_2$). Assuming the original frame of the events is our rest frame we have the following Lorentz transformation for the time in a frame $\bar{O}$ moving with velocity v with respect to the rest frame.

$$\bar{t} = \gamma\left(t - \frac{vx}{c^2}\right)$$

Requiring that our two events be simultaneous in this $\bar{O}$ frame, we find

$$\begin{aligned}
\bar{t}_1 &= \bar{t}_2 \\
\gamma\left(t_1 - \frac{vx_1}{c^2}\right) &= \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \\
\gamma\left(0 - \frac{v\cdot 0}{c^2}\right) &= \gamma\left(\frac{1}{c} - \frac{2v}{c^2}\right) \\
0 &= \left(\frac{1}{c} - \frac{2v}{c^2}\right) \\
\frac{1}{c} &= \frac{2v}{c^2} \quad \Longrightarrow \quad \boxed{v = c/2}
\end{aligned}$$

So the $\bar{O}$ frame moves away from the rest frame with relative velocity $c/2$ in the x direction.

Problems

1. **Problem 1. A Passing Stick: A stick of length L moves past you at speed v. There is a time interval between the front end coinciding with you and the back end coinciding with you. What is this time interval in**
   (a) your frame? (Calculate this by working in your frame)
   (b) your frame? (Work this in the stick's frame)
   (c) the stick's frame? (Work in your frame. This is tricky).
   (d) the stick's frame? (Work in the stick's frame)

   Solution: (a) In our frame, if a stick of rest length L is moving at a velocity v then the stick is length contracted down to L/γ. Consequently, the time interval between the front end coinciding with us and the back end coinciding with us is $\boxed{L/\gamma v}$.

   (b) In the stick's frame it has a length L and we are moving at a velocity v towards it. So, according to the stick, a time L/v passes as we move from one end of the stick to the other. In the stick's reference frame, we are moving so time must run more slowly for us than for the stick. So, if the stick experiences a time L/v we must experience a time $\boxed{L/\gamma v}$.

   (c) From the rear clock ahead effect, we know that the rear end of the stick measures a set time ahead of the front end. So, when the front end of the stick reaches us, this front end reads $t = 0$ and the back end reads $t = Lv/c^2$. In our frame, the stick is length contracted down to L/γ so that the time it takes to pass the length of the stick, from front to back end, is $L/\gamma v$ in the ground frame. This time is therefore $L/\gamma^2 v$ in the stick's frame. So, if we start at the front end of the stick with

$t = 0$ and travel to the back end of the stick, the time at this back end would read

$$\frac{L}{\gamma^2 v} + \frac{Lv}{c^2} = \frac{L}{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \frac{L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \boxed{\frac{L}{v}}$$

where we added the time of travel and set time that the back end is ahead of the front end.

(d) As we found in (b), the time that it takes us to pass between the two ends of the stick is $\boxed{L/v}$ in the stick's frame.

2. **Problem 2. Rotated Square: A square with side L flies past you at speed v, in a direction parallel to two of its sides. You stand in the plane of the square. When you see the square at its nearest point to you, show that it looks to you like it is rotated, instead of contracted. Assume that L is small compared with the distance between you and the square. Solution:**

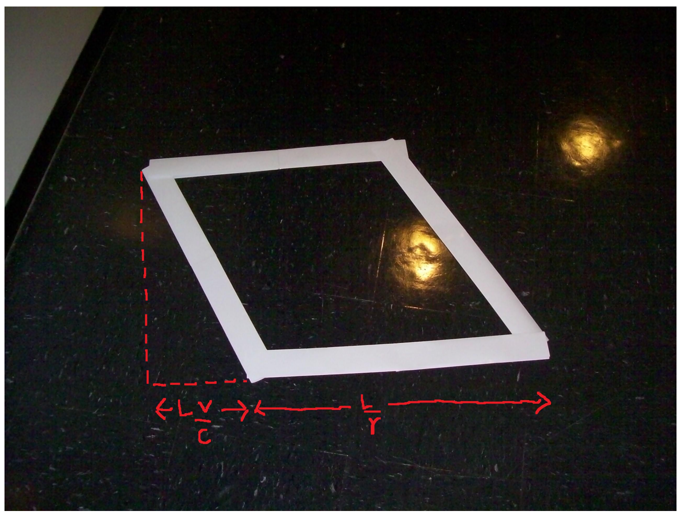

Figure 1: Parallelogram from close up

Imagine that we are standing in the plane of the square which is moving to the right with velocity v. From the fundamental effects of special relativity, we automatically know that the square is length contracted so that it becomes a rectangle with the short side parallel to the direction of motion. A rectangle is what the square *becomes* but the question is what does the square *look like*? More specifically, where are different parts of the square when the light from the square hits your eye? The answer has to do with the signal propagation of the photons. Assume we are standing in front of one of the length contracted sides of the square. Then, the photons from the opposite length contracted side must travel an extra distance L compared to the photons from the near side to reach our eyes. Consequently, the opposite side photons require an extra time L/c of flight. But as these opposite side photons travel towards our eyes, the square moves to the right a certain distance. Specifically, when these opposite side photons traverse the length L of the square, the square has already moved

Figure 2: Parallelogram from far away (looks like rotated square from far away)

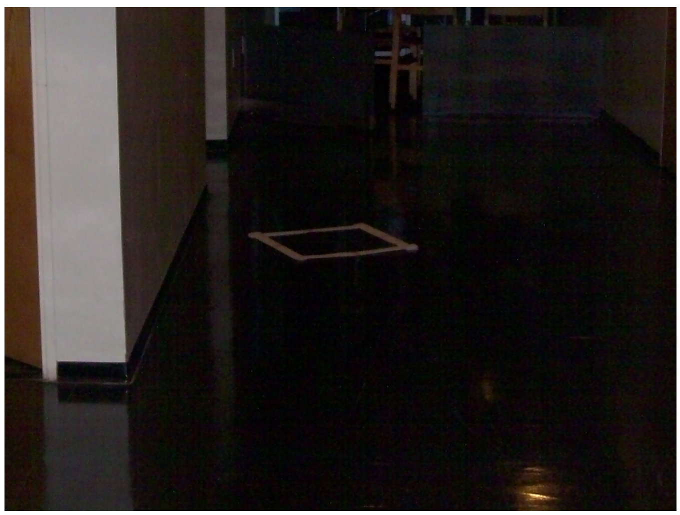

Figure 3: Rotated square from far away (looks like parallelogram from far away)

a distance Lv/c to the right. At this point in time, the near side photons can be emitted from the square and they will reach our eyes at the same time that these delayed opposite side photons do. The result is that we observe the opposite side of square as *farther to the left* than the near side which results in a quadrilateral which no longer looks like a square. The quadrilateral looks like the one in Fig.1 and given the requirement that we are far away from the object it looks like Fig.2. At first glance, this looks nothing like a rotated square. But, when we consider Fig.3, a rotated square from far away, and Fig.4, the same square from close up, we can understand why the problem makes the requirement that we are standing far away from the square. From this far away position, the object that looks like a parallelogram also looks like a rotated square.

3. **Problem 3. Cookie Cutter: Cookie dough lies on a conveyor belt that moves at speed v. A circular stamp stamps out cookies as the dough rushes by beneath it. When you buy these cookies in a store, what shape are they? That is, are they squashed in the**

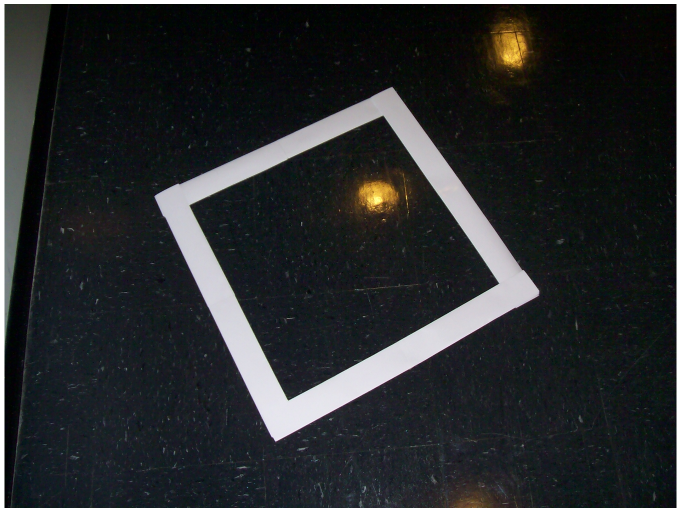

Figure 4: Rotated square from close up

direction of the belt, stretched in that direction, or circular? Carefully show, step by step, your reasoning.

Solution: It is easiest to understand this problem from the frame in which we buy the cookies. This is the frame which ultimately matters in the end and is the same frame as the ground frame. From this frame, the dough rushes by on the conveyor belt and becomes length contracted. Assuming that the diameter of the cookie cutter is L when the cookies are cut, then the cookies will have a length of $L\gamma$ at rest because this length is what is length contracted down to the cookie cutter diameter L. So, when we buy the cookies they look $\boxed{\text{stretched out in the direction of the conveyor belt.}}$
When we consider the same problem from the perspective of the cookie we obtain seemingly contradictory results, because from this perspective the cookie cutter is the one that seems length contracted. However, because of the loss of simultaneity, the simultaneous cutting which occurs in the ground frame does not occur in the cookie's frame. In fact this problem uses the same analysis as in the Length Contraction Paradox of the first recitation's Relativity Notes to conclude that the final length of the cookies is γL. Specifically from the dough's frame, the right most point of the cutter reaches the dough at one point in time and the leftmost point reaches the dough a time $\gamma Lv/c^2$ later where the factor of γ comes from a consideration of time dilation. The cutter therefore travels an extra distance $\gamma Lv^2/c^2$ in addition to its length L/γ. So we find that the total length of the resulting cookie is

$$\frac{L}{\gamma} + \frac{L\gamma v^2}{c^2} = L\gamma\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = L\gamma\left(\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right) = L\gamma$$

4. **Problem 4. Train in a Tunnel and a Bomb: A train and a tunnel both have rest lengths L. The train moves toward the tunnel at speed v. A bomb is located at the front of the train. The bomb is designed to explode when the front of the train passes the far end of the tunnel. A deactivation sensor is located at the back of the train. When the back of the train passes the rear end of the tunnel, the sensor tells the bomb to disarm itself. Does the bomb explode? Carefully show, step-by-step, your reasoning. Simply stating "Yes" or "No" will get you no points on this problem.**

   Solution: $\boxed{\text{The bomb explodes}}$. We can see this most easily by considering the problem from the train's frame. In the train's frame the length of the tunnel is contracted so that it becomes shorter than the train's length. Therefore, the front end of the train reaches the far end of the tunnel while the back end of the train is still outside the tunnel and the bombs must go off without being deactivated. The paradox comes from considering the same situation from the tunnel's frame in which the train is now shorter; the back end of the train reaches the near side of the tunnel before the front end of the train reaches the far end of the tunnel and the bomb must therefore be deactivated before going off. The resolution of the paradox comes from the fact that the signal that the back end of the train sends to the front end is not able to reach the front end before the bombs go off. Let t_L be defined as the time that the signal takes to get to the front end of the train and let t_T be the time that it takes the front end of the train to reach the far end of the tunnel after the deactivation signal goes off. So, assuming that this signal travels at the maximum speed possible, c, t_L must satisfy

$$\frac{L}{\gamma} + vt_L = ct_L \qquad \Longrightarrow \qquad t_L = \frac{L/\gamma}{c-v}$$

where we include a factor of $1/\gamma$ because we are observing the moving train from the ground frame. Next, we know that t_T must satisfy

$$\frac{L - L/\gamma}{v} = t_T$$

Now, we must prove that $t_T < t_L$ for all v even when the signal travels at the speed of light c. So, we have

$$\begin{aligned}
t_T &< t_L \\
\frac{L - L/\gamma}{v} &< \frac{L/\gamma}{c-v} \\
\frac{1 - 1/\gamma}{v} &< \frac{1/\gamma}{c-v} \\
\frac{1/\gamma - 1/\gamma^2}{v} &< \frac{1/\gamma^2}{c(1-v/c)} \\
&= \frac{1 - v^2/c^2}{c(1-v/c)} \\
&= \frac{1 + v/c}{c} \\
\frac{1/\gamma - (1 - v^2/c^2)}{v} &< \frac{1}{c} + \frac{v}{c^2} \\
\frac{1}{\gamma v} - \frac{1}{v} + \frac{v}{c^2} &< \frac{1}{c} + \frac{v}{c^2} \\
\frac{1}{\gamma v} &< \frac{1}{v} + \frac{1}{c} \\
\frac{1}{v}\sqrt{1 - v^2/c^2} &< \frac{1 + v/c}{v} \\
\sqrt{1 - v^2/c^2} &< (1 + v/c) \\
\sqrt{1 - v/c} &< \sqrt{1 + v/c}
\end{aligned}$$

The last line is always true for $v > 0$ so $t_T < t_L$ is true and the train always reaches the end of the tunnel before the light signal reaches the end of the train. So, the bomb always explodes.

5. **Problem 5. Clapping Both Ways: Twin A stays on earth, and twin B flies to a distant star and back. (a) Throughout the trip, B claps in such a way that his claps occur at**

equal time intervals Δt in A's frame. At what time intervals do the claps occur in B's frame?
(b) Now, let A clap in such a way that his claps occur at equal time intervals Δt in B's frame. At what time intervals do the claps occur in A's frame? (This is tricky. Note that the sum of all the time intervals must equal the increase in A's age, which is greater than the increase in B's age, in accordance with the twin paradox).

Solution: (a) From A's perspective, B's clock must run slow because B is moving. So in order for the claps to occur at equal time intervals Δt in A's frame, these claps must occur at time intervals $\boxed{\Delta t/\gamma}$ in B's frame.

(b) It appears as though by applying the same logic as in (a) we should get the same answer: A should clap at intervals $\Delta t/\gamma$ in order for the time intervals to be Δt in B's frame. But, this answer contradicts the standard Twin Paradox Result in that it implicitly states that more time passes in B's frame than in A's frame. The solution to this paradox comes from a consideration of the Rear Clock Ahead Effect. Let the distance between the star and earth be L. Then from B's perspective, the two planets are a distance L/γ apart and are traveling towards it with speed v. Also we know that when Earth's clock reads $t = 0$, the star's clock must read $t = \frac{Lv}{c^2}$ because in this frame the star is at the rear of a moving length. This is all fine until we consider the return trip. When B reverses direction, then all of a sudden Earth's clock is the rear clock and must therefore be ahead of the star's clock by a time Lv/c^2. For example, if we let $Lv/c^2 = 5$min and we get to the star at 5:05 pm (star's time) then Earth's clock must show 5:00 pm, because the star's clock is ahead. But, when we reverse directions, Earth's clock must be ahead so it must change from 5:00pm to 5:10pm. In effect, it appears as though Earth's clock jumped from 5:00 to 5:10 in an instant. Consequently anything that happened on earth in this 10 minute period, is not observed by B. This does not occur in (a) because in that case A is looking at the single clock in B's frame and the rear clock ahead effect is not present for signle points in space.
From this example, we realize that the correct behavior for A is to clap at a rate $\Delta t/\gamma$ for some time period and then stop clapping for a time $2Lv/c^2$ and then resume clapping at $\Delta t/\gamma$. Specifically, A must clap for as long as B is moving and before B changes direction. Since B believes it travels for a time $L/\gamma v$ between the planets A must observe this time as $L/\gamma^2 v$ and this is therefore the amount of time that A must clap.

$$\boxed{\begin{array}{l}A \text{ claps at intervals } \Delta t/\gamma \text{ for a time } L/\gamma^2 v \text{ and then stops for a time } 2Lv/c^2 \text{ and then continues} \\ \text{to clap at intervals } \Delta t/\gamma \text{ for another time interval } L/\gamma^2 v\end{array}}.$$

We notice that the sum of these times produces the expected result

$$\frac{2L}{\gamma^2 v} + \frac{2vL}{c^2} = \frac{2L}{v}\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \frac{2L}{v}\left(\frac{v^2}{c^2} + \left(1 - \frac{v^2}{c^2}\right)\right) = \frac{2L}{v}$$

which is the total time A measures as B travels from Earth and Back.

6. **Problem 6. Many velocity additions: An object moves at speed $c\beta_1$ with respect to observer O_1, which moves at speed $c\beta 2$ with respect to O_2, which moves at speed $c\beta_3$ with respect to O_3 and so on, until finally O_{N-1} moves at speed $c\beta_N$ with respect to O_N. Show by mathematical induction that the speed, call it $c\beta_{(N)}$ of the object with respect to O_N can be written as $\beta_{(N)} = \dfrac{P_N^+ - P_N^-}{P_N^+ + P_N^-}$**

where $P_N^+ \equiv \prod_{i=1}^{N}(1+\beta_i)$ **and** $P_N^+ \equiv \prod_{i=1}^{N}(1-\beta_i)$**.**

Solution: The induction procedure is composed of the following steps
1)Show that the formula is true for $N = 1$
2)Assume the formula is true for k
3)Show using the formula in step 2) that the formula is true for $k + 1$

First checking our result for $N = 1$

$$\beta_{(1)} = \frac{P_1^+ - P_1^-}{P_1^+ + P_1^-} = \frac{(1+\beta_1) - (1-\beta_1)}{(1+\beta_1) + (1-\beta_1)} = \beta_1$$

This translates into the statement that the relative velocity of the object with respect to O_1 is β_1. This result is obvious from the problem statement, so the first condition is satisfied.
Now we assume that the formula for $\beta_{(k)}$ is generally true.

$$\beta_{(k)} = \frac{P_k^+ - P_k^-}{P_k^+ + P_k^-}$$

and from this assumption we must show that $\beta_{(k+1)}$ is true. In particular we must show that $\beta_{(k+1)}$ is equivalent to the standard result from relativity. The speed $\beta_{(k+1)}$ represents the speed of the object with respect to O_{k+1} which is the result of relativistically adding the speed of the object with respect to O_k (which is $\beta_{(k)}$) with the speed of O_k with respect to O_{k+1} (which is β_{k+1}). Therefore $\beta_{(k+1)}$ is

$$\beta_{(k+1)} = \frac{\beta_{k+1} + \beta_{(k)}}{1 + \beta_{k+1}\beta_{(k)}}$$

And assuming that the formula is true for $N = k$, we have

$$\begin{aligned}
\beta_{k+1} = \frac{\beta_{k+1} + \frac{P_k^+ - P_k^-}{P_k^+ + P_k^-}}{1 + \beta_{k+1}\frac{P_k^+ - P_k^-}{P_k^+ + P_k^-}} &= \frac{\beta_{k+1}(P_k^+ + P_k^-) + (P_k^+ - P_k^-)}{(P_k^+ + P_k^-) + \beta_{k+1}(P_k^+ - P_k^-)} \\
&= \frac{(1+\beta_{k+1})P_k^+ - P_k^-(1-\beta_{k+1})}{(1+\beta_{k+1})P_k^+ + P_k^-(1-\beta_{k+1})} \\
&= \frac{(1+\beta_{k+1})\prod_{i=1}^{k}(1+\beta_i) - (1-\beta_{k+1})\prod_{i=1}^{k}(1-\beta_i)}{(1+\beta_{k+1})\prod_{i=1}^{k}(1+\beta_i) + (1-\beta_{k+1})\prod_{i=1}^{k}(1-\beta_i)} \\
&= \frac{\prod_{i=1}^{k+1}(1+\beta_i) - \prod_{i=1}^{k+1}(1-\beta_i)}{\prod_{i=1}^{k+1}(1+\beta_i) + \prod_{i=1}^{k+1}(1-\beta_i)} \\
&= \frac{P_{k+1}^+ - P_{k+1}^-}{P_{k+1}^+ + P_{k+1}^-}
\end{aligned}$$

So the formula is true for $\beta_{(k+1)}$ and the induction is complete. Alternatively, we could have gone in the reverse direction by beginning with this last line as our first step in the $(k+1)$ induction and then going on to show that the formula produces the standard velocity addition result.

Now for some mathematics: Why does induction work? Well, we first prove that the rule is true for 1. Then we prove that if the rule is true for general k then the rule must be true for $k+1$. Therefore, if it is true for 1 it must be true for 1+1=2, and if it's true for 2 it must be true for 3 and so on. Thus, we find that the property is true for all positive integers.