MITE^{2}S 2010: Physics III
Survey of Modern Physics
Problem Set 2 Solutions

1. **Problem 1. What a drag!: Suppose a particle moves in one-dimension (i.e., along x-axis). $x(t)$ is the particles position at time t. A drag force proportional to the particles velocity v acts on it (i.e. Drag force on the particle is $f_{\mathbf{drag}} = -bv$). Write down the equation of the motion of the particle. Then by solving this equation, derive the velocity $v(t)$ and the position $x(t)$ of the particle. If there are any free parameters in your solution, describe what they physically mean. (If you get stuck on an integration, you can look up the formula in a book / web).**

   Solution: The only force in this problem is the drag force $-bv$ so that $F_{\text{NET}} = -bv$ and with Newton's Second Law we have

$$\begin{aligned} F_{\text{NET}} &= ma \\ -bv &= m\frac{dv}{dt} \end{aligned}$$

   which, when rearranged, gives us the EQUATION OF MOTION (EOM).

$$\boxed{\frac{dv}{dt} + \frac{b}{m}v = 0}$$

   To solve this equation for $v(t)$ we use the technique of separation of variables

$$\begin{aligned} \frac{dv}{dt} &= -\frac{b}{m}v \\ \frac{dv}{v} &= -\frac{b}{m}dt \\ \int_{v_0}^{v(t)} \frac{dv}{v} &= -\int_0^t \frac{b}{m}dt' \\ \ln\left[\frac{v(t)}{v_0}\right] &= -\frac{bt}{m} \\ \frac{v(t)}{v_0} &= e^{-\frac{bt}{m}} \end{aligned}$$

   and we have

$$\boxed{v(t) = v_0 e^{-\frac{bt}{m}}}$$

   In the third line, we chose the limits for the velocity definite integral to correspond to the limits for the time integral. So v_0 represents the velocity at time $t' = 0$ (i.e. the initial velocity) and $v(t)$ represents the velocity at some arbitray time $t' = t$. With this result for velocity we obtain another differential equation which we may use to solve for $x(t)$.

$$\begin{aligned} v(t) = \frac{dx}{dt} &= v_0 e^{-\frac{bt}{m}} \\ dx &= v_0 e^{-\frac{bt}{m}} dt \\ \int_{x_0}^{x(t)} dx &= \int_0^t v_0 e^{-\frac{bt'}{m}} dt' \\ x(t) - x_0 &= -\frac{mv_0}{b} \left[e^{-\frac{bt}{m}} \right]_0^t \\ &= \frac{mv_0}{b} \left(1 - e^{-\frac{bt}{m}} \right) \end{aligned}$$

So $x(t)$ is

$$\boxed{x(t) = x_0 + \frac{mv_0}{b} \left(1 - e^{-\frac{bt}{m}} \right)}$$

Like in the velocity case, we chose the limits of our position integral to correspond to the limits of the time integral. So x_0 represents the position at time $t' = 0$ (i.e. the initial position) and $x(t)$ represents the position at some arbitrary time t.
In summary, x_0 and v_0 represent the **initial conditions** of our problem. Whenever we solve Newton's Second Law, which is a second order differential equation in time, we must always include these two free parameters which define our system at the beginning of the relevant motion.

2. **Problem 2. Optical Tweezer: Optical tweezer is a device that traps dielectric microspheres using a finely focused laser beam. By sending a beam with just the right intensity profile, the microsphere (bead) can be trapped within a simple harmonic potential well formed by the laser beam, with the center of the beam being the minimum of the potential well. That is, if the bead is displaced by x from the trap center, a restoring force of $-k_{\mathbf{trap}}x$ acts on the bead in an attempt to return it to the trap center ($x = 0$). We can also attach polymers such as DNA to the bead, with one end of the DNA tethered to a stationary glass slide. As long as we are dealing with small stretch and compression of DNA, we can model DNA as a Hookian spring-like object with an effective spring constant $k_{\mathbf{DNA}}$. Thus, the system can be modeled as a bead of mass m attached to two springs (with spring constants $k_{\mathbf{DNA}}$ and $k_{\mathbf{trap}}$).**

   (a) Derive the equation of motion for the bead (shown in Fig. 1c) and thus show that the bead acts as a simple harmonic oscillator. State what the angular frequency ω is, in terms of $k_{\mathbf{DNA}}$, $k_{\mathbf{trap}}$, and m. You can assume that $x = 0$ is the equilibrium position of the bead and that both springs are at their rest lengths at $x = 0$.

   Solution: The bead is attached to two springs at two opposite ends. If we perturb the bead from its equilibrium position we can find the forces which act on it. For example, if we move the bead to the right a distance x then there is a force pulling to the left coming from the spring with constant k_{trap}. This force opposes the positive increase in position so we may write it as $F_{\text{trap}} = -k_{\text{trap}}x$. The bead also feels a force pushing to the left coming from the spring with constant k_{DNA}. This force opposes the positive increase in position so we may write it as $F_{\text{DNA}} = -k_{\text{DNA}}x$. Therefore, we find that the net force is

$$F_{NET} = F_{\text{DNA}} + F_{\text{trap}} = -k_{\text{DNA}}x - k_{\text{trap}}x$$

and the resulting EOM is

$$\begin{aligned} F_{NET} &= m\ddot{x} \\ -k_{\text{DNA}}x - k_{\text{trap}}x &= m\ddot{x} \\ 0 &= m\ddot{x} + (k_{\text{DNA}} + k_{\text{trap}})x \qquad \Longrightarrow \qquad \boxed{0 = \ddot{x} + \omega^2 x} \end{aligned}$$

where we have defined $\boxed{\omega = \sqrt{\dfrac{k_{\text{trap}} + k_{\text{DNA}}}{m}}}$

(b) Derive the general solution to the equation of motion you derived in (a). Define the physical meaning of all the free parameters in your solution.

Solution: The general solution to any simple harmonic oscillator EOM (i.e. $\ddot{x} + \omega^2 x = 0$) is

$$x(t) = A \sin \omega t + B \cos \omega t$$

In order to find what A and B represent we solve for the initial conditions of our bead. Setting $t = 0$ in our position equation, we find

$$x(t = 0) = A \sin 0 + B \cos 0 = B \qquad \Longrightarrow \boxed{B = x(t = 0)}$$

B represents the initial position of the bead. Similarly, differentiating $x(t)$ and then plugging in $t = 0$ gives us

$$v(t = 0) = \dot{x}(t = 0) = A\omega \cos 0 - B\omega \sin 0 = A\omega \qquad \Longrightarrow \boxed{A = \frac{v(t = 0)}{\omega}}$$

A represents the inital velocity of the bead divided by the angular frequency. All of this information gives us a general solution $x(t)$ of the form

$$x(t) = \frac{v(t = 0)}{\omega} \sin \omega t + x(t = 0) \cos \omega t$$

(c) Write down the total energy of the system shown in Fig. 1(c). At what value(s) of x is the kinetic energy maximum? At what value(s) of x is the potential energy of DNA spring maximum? At what value(s) of x is the potential energy of the trap maximum? Answer in terms of one of the free parameters you defined in (b). Also, what is the ratio of potential energy stored in the trap-spring to the potential energy stored in the DNA spring at time t?

Solution: Total energy is defined as

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

For this problem we know that the force $F(x)$ which acts on the bead is $F(x) = -(k_{\text{trap}} + k_{\text{DNA}})x$. Using this equation for force we can find the potential energy $V(x)$ for an arbitrary position x.

$$\begin{aligned} V(x) &= -\int_0^x F(x')\,dx' \\ &= -\int_0^x (-)(k_{\text{trap}} + k_{\text{DNA}})x'\,dx' \\ &= (k_{\text{trap}} + k_{\text{DNA}})\int_0^x x'\,dx' \\ &= \frac{(k_{\text{trap}} + k_{\text{DNA}})x^2}{2} \end{aligned}$$

So the total energy of the system is $\boxed{E_{\text{Total}} = \frac{1}{2}m\dot{x}^2 \frac{(k_{\text{trap}} + k_{\text{DNA}})x^2}{2}}$.

From our equation for energy and with the fact that the value of energy is always the same, we know that kinetic energy is maximum when potential energy is zero and potential energy is maximum when kinetic energy is zero. Treating the first case, we know that potential energy is zero when the spring is neither streched or compressed, that is when it is at its equilibrium position $x = 0$. So, $\boxed{\text{KE is max when } x = 0}$. The second case is not as simple. We know that kinetic energy is zero when velocity is zero, but when is velocity zero? We can solve for the point/time that velocity is zero by using our general solution for $x(t)$.

$$\begin{aligned} 0 = \dot{x(t)} &= -A\omega\cos\omega t + B\omega\sin\omega t \\ \tan\omega t &= \frac{A}{B} \end{aligned}$$

So velocity is zero when t satisfies the last line above. It turns out that this time represents how long it takes the bead to reach its **turning point** where it changes from positive to negative velocity (or vice versa). The last line of the above equation also implies similar relations for $\sin\omega t$ and $\cos\omega t$. They are

$$\sin\omega t = \frac{A}{\sqrt{A^2 + B^2}} \qquad \cos\omega t = \frac{B}{\sqrt{A^2 + B^2}}$$

with these relations we can find the point x_{MAX} where potential energy is maximum

$$\begin{aligned} x_{\text{MAX}} &= A\frac{A}{\sqrt{A^2 + B^2}} + B\frac{B}{\sqrt{A^2 + B^2}} \\ &= \frac{A^2 + B^2}{\sqrt{A^2 + B^2}} \\ &= \sqrt{A^2 + B^2} \end{aligned}$$

Alternatively, $\sin\omega t$ and $\cos\omega t$ could be the negative of the values we chose for them and still reproduce the $\tan\omega t = \frac{A}{B}$ result. Considering this "negative case"', we find that another value of x_{MAX} is $x_{\text{MAX}} = -\sqrt{A^2 + B^2}$. So in general we see that $\boxed{\text{PE is max when } x = \pm\sqrt{A^2 + B^2}}$[1]. We found a value of x_{MAX} which is independent of ω and consequently independet of k. So this value of x represents the position where potential energy is maximum in both the trap and DNA case.
The ratio between the potential energy of the trap spring and the DNA spring is easily found by using their respective formulas for potential energy

[1] If we used E and ϕ as our free parameters this result would be $x = \pm E$

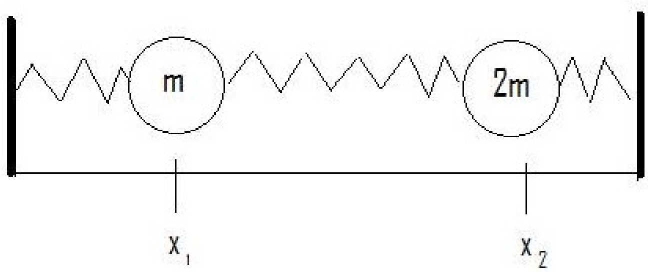
$$\frac{\text{PE}_{trap}}{\text{PE}_{DNA}} = \frac{\frac{1}{2}k_{\text{trap}}x(t)^2}{\frac{1}{2}k_{\text{DNA}}x(t)^2} = \boxed{\frac{k_{\text{trap}}}{k_{\text{DNA}}}}$$

(d) Suppose that at $t = 0$, the bead is at $x(t = 0) = 0$, and that its velocity at that moment is v_0. By finding specific values for the free parameters in your solution $x(t)$ found in (b), write down $x(t)$ that describes what the position of the bead is for subsequent times ($t > 0$).

Solution: From (a), we know that with the initial conditions $x(t = 0) = 0$ and $v(t = 0) = v_0$ we must have $A = v_0/\omega$ and $B = 0$. Therefoe our $x(t)$ becomes

$$\boxed{x(t) = \frac{v_0}{\omega}\sin\omega t}$$

3. **Problem 3. Normal modes I: Three identical springs and two masses, m ande $2m$, lie between two walls. Find the normal modes and write down the general solution describing any arbitrary motion that is executed by the two particles.**



Solution: The net force on each block is independent of the mass of the block, so the equations for force in this case are identical to the ones we derived in the class for the equal mass case

$$\begin{aligned} F_1 &= -2kx_1 + kx_2 \\ F_2 &= kx_1 - 2kx_2 \end{aligned}$$

Here x_1 denotes the position of the block with mass m and x_2 denotes the position of the block with mass $2m$. From these forces and Newton's Second Law we obtain the following EOMs.
For x_1

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ \ddot{x}_1 &= -2\omega^2 x_1 + \omega^2 x_2 \end{aligned}$$

and then for x_2

$$\begin{aligned} 2m\ddot{x}_2 &= kx_1 - 2kx_2 \\ \ddot{x}_2 &= \frac{1}{2}\omega^2 x_1 - \omega^2 x_2 \end{aligned}$$

where in the second line of each step, we defined $\omega = \sqrt{k/m}$. Collecting, these two EOMs into a system of equations we can represent them in matrix form.

$$\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -2\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{1}$$

To solve this system for x_1 and x_2, we guess an arbitrary solution and adjust the various parameters in our guess to conform to the EOMs. Let our guess be

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}$$

so that the right hand side of Eq (1) becomes

$$\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} = \alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t}$$

Plugging our guess and its second derivative into our original equation of motion we find

$$\begin{aligned}
\begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} &= \begin{pmatrix} -2\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\
\alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t} &= \begin{pmatrix} -2\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} e^{\alpha t} \\
0 &= \begin{pmatrix} -2\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} - \alpha^2 \begin{pmatrix} A \\ B \end{pmatrix} \\
&= \begin{pmatrix} -2\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} - \alpha^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
0 &= \begin{pmatrix} -2\omega^2 - \alpha^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 - \alpha^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
0 &= \hat{M}\vec{a}
\end{aligned}$$

In the fourth line, we multiplied the 2×1 matrix by the 2×2 identity matrix in order to be able to add it to the frequency matrix. In the last line, we defined $\vec{a}$ as the 2×1 coefficient matrix and $\hat{M}$ as the frequency matrix.
Now, we are looking for A, B, and α and we know that in order to obtain a nontrivial solution the inverse of $\hat{M}$ cannot exist. So the $\det\hat{M}$ must equal zero and we must choose α such that this condition is satisfied. Calculating the determinant and setting it equal to zero, we have

$$\begin{aligned}
0 &= \begin{vmatrix} -2\omega^2 - \alpha^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 - \alpha^2 \end{vmatrix} \\
&= (-2\omega^2 - \alpha^2)(-\omega^2 - \alpha^2) - \frac{\omega^4}{2} \\
0 &= \frac{3}{2}\omega^4 + 3\alpha^2\omega^2 + \alpha^4 \\
0 &= 3\omega^4 + 6\omega^2\alpha^2 + 2\alpha^4
\end{aligned}$$

Using the quadratic formula to solve for α^2, we find the following two roots

$$\alpha_1^2 = \frac{-(3-\sqrt{3})}{2}\omega^2 \qquad\qquad \alpha_2^2 = \frac{-(3+\sqrt{3})}{2}\omega^2$$

these roots define the characteristic frequencies of the system similar to the way k/m defines the frequency in a simple harmonic oscillator system. Each α^2 defines a particular set of values for A and B, and by plugging in each characteristic frequency we can find the associated coefficients.

For the $\alpha_1^2 = \dfrac{-(3-\sqrt{3})}{2}\omega^2$ case:

$$
\begin{aligned}
0 &= \hat{M}_1 \vec{a}_1 \\
&= \begin{pmatrix} -2\omega^2 + \frac{(3-\sqrt{3})}{2}\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 + \frac{(3-\sqrt{3})}{2}\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
&= \omega^2 \begin{pmatrix} \frac{(-1-\sqrt{3})}{2} & 1 \\ \frac{1}{2} & \frac{(1-\sqrt{3})}{2}\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\end{aligned}
$$

Solving for A and B in this equation, we find that[2] $B_1 = A_1 \dfrac{(1+\sqrt{3})}{2}$ and

$$
\vec{a}_1 = A_1 \begin{pmatrix} 1 \\ \frac{(1+\sqrt{3})}{2} \end{pmatrix}
$$

Similarly for the $\alpha_2^2 = \dfrac{-(3+\sqrt{3})}{2}\omega^2$ case:

$$
\begin{aligned}
0 &= \hat{M}_2 \vec{a}_2 \\
&= \begin{pmatrix} -2\omega^2 + \frac{(3+\sqrt{3})}{2}\omega^2 & \omega^2 \\ \frac{1}{2}\omega^2 & -\omega^2 + \frac{(3+\sqrt{3})}{2}\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \\
&= \omega^2 \begin{pmatrix} \frac{(-1+\sqrt{3})}{2} & 1 \\ \frac{1}{2} & \frac{(1+\sqrt{3})}{2}\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\end{aligned}
$$

So we find that $B_2 = A_2 \dfrac{(1-\sqrt{3})}{2}$ and

$$
\vec{a}_2 = A_2 \begin{pmatrix} 1 \\ \frac{(1-\sqrt{3})}{2} \end{pmatrix}
$$

For each $\vec{a}$ we know that there is one associated value of α^2 and therfore two associated valus of α (i.e. $\pm\sqrt{\alpha^2}$). In total this gives us four exponential solutions which when added together give us the most general solution. This most general solution is

$$
\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ \frac{(1+\sqrt{3})}{2} \end{pmatrix} \left(A_{1,+} \exp i\omega\sqrt{\frac{3+\sqrt{3}}{2}} + A_{1,-} \exp -i\omega\sqrt{\frac{3+\sqrt{3}}{2}} \right) \\
&+ \begin{pmatrix} 1 \\ \frac{(1-\sqrt{3})}{2} \end{pmatrix} \left(A_{2,+} \exp i\omega\sqrt{\frac{3-\sqrt{3}}{2}} + A_{2,-} \exp -i\omega\sqrt{\frac{3-\sqrt{3}}{2}} \right)
\end{aligned}
$$

[2] It seems as though we get two inconsistent equations for A and B, but a little algebra can show that the equations produce the same result.

where $\exp[x] = e^x$. We can write this result more compactly by using one of our alternate formulas for the solution to the simple harmonic oscillator. The result is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ \frac{(1+\sqrt{3})}{2} \end{pmatrix} \left[\cos\left(\omega\sqrt{\frac{3+\sqrt{3}}{2}} - \phi_1\right)\right] + C_2 \begin{pmatrix} 1 \\ \frac{(1-\sqrt{3})}{2} \end{pmatrix} \left[\cos\left(\omega\sqrt{\frac{3-\sqrt{3}}{2}} - \phi_2\right)\right]}$$

This represents the most general solution and each term in this two term sum is one of the **normal modes** of the system.

4. **Problem 4. Heading to zero: A particle moves toward $x = 0$ under the influence of a potential $V(x) = -Ax^n$, where $A > 0$ and $n > 0$. The particle has barely enough energy to reach $x = 0$. For what values of n will it reach $x = 0$ in a finite time?**

   Solution: The energy of the particle is $E = mv^2/2 - A|x|^n$. If the particle has barely enough energy to reach $x = 0$ then the particle would have a kinetic energy which is nearly zero at this point. So, we find that $E = 0 - A|0|^n = 0$ at the end point. Energy is conserved, so the total energy must be zero at all points along the trajectory. That is

$$0 = E = mv^2/2 - A|x|^n \qquad \Longrightarrow \qquad v = -\sqrt{2Ax^n/m}$$

   where we chose the negative root for v because we are letting $x > 0$ which makes a particle traveling to the origin have negative velocity. Writing v as dx/dt we have a differential equation for x

$$\frac{dx}{dt} = -\sqrt{2Ax^n/m}$$

   Which when solved by separation of variables, yields

$$\begin{aligned} -\sqrt{2Ax^n/m} &= \frac{dx}{dt} \\ -\sqrt{\frac{2A}{m}}dt &= x^{-n/2}\,dx \\ -\sqrt{\frac{2A}{m}}\int_0^T dt &= \int_{x_0}^0 x^{-n/2}dx \\ -T\sqrt{\frac{2A}{m}} &= \left[\frac{x^{(1-n/2)}}{1-n/2}\right]_{x_0}^0 \\ &= \frac{0^{(1-n/2)}}{1-n/2} - \frac{x_0^{(1-n/2)}}{1-n/2} \end{aligned}$$

   From the last line, we see that the right hand side, which is a constant times time T, is finite only if $1 - n/2 > 0$ or $\boxed{2 > n}$.

5. **Problem 5. Hanging mass: The potential energy for a mass hanging from a spring is $V(y) = ky^2/2+mgy$, where $y = 0$ corresponds to the position of the spring when nothing is hanging from it. Find the frequency of small oscillations around the equilibrium point.**

Solution: We will calculate the frequency of oscillations with the formula

$$\omega = \sqrt{\frac{1}{m} V''(y_0)} \tag{2}$$

where y_0 is the equilibrium position defined by $V'(y_0) = 0$. Solving for this position y_0 we find

$$0 = V'(y) \quad = \quad ky + mg \quad \Longrightarrow \quad \boxed{y_0 = -\frac{mg}{k}}$$

However, by calculating the second derivative of $V(y)$ we see that the equilibrium position is not necesseary to define the oscillation frequency.

$$V''(y) = k$$

So that

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

6. **Problem 6. Small oscillation: A particle moves under the influence of the potential $V(x) = -Cx^n e^{-ax}$. Find the frequency of small oscillations around the equilibrium point.**

   Solution: The frequency of oscillations about the equilibrium position x_0 is defined as

$$\omega = \sqrt{\frac{1}{m} V''(x_0)}$$

where $V''(x_0)$ is the second derivative of $V(x)$ evaluated at the point x_0. This point x_0 is defined by the fact that $V'(x_0) = 0$. Solving for this point we find

$$\begin{aligned}
0 = V'(x) \quad &= \quad \frac{d}{dx}(-Cx^n e^{-ax}) \\
0 \quad &= \quad -Cnx^{n-1}e^{-ax} + Cax^n e^{-ax} \\
&= \quad -nx^{n-1} + ax^n \\
&= \quad -n + ax \quad \Longrightarrow \quad \boxed{x_0 = \frac{n}{a}}
\end{aligned}$$

and then calculating the second derivative and plugging in this equlibrium point

$$
\begin{aligned}
V''(x) &= \frac{d}{dx}[(-nx^{n-1}+ax^n)Ce^{-ax}] \\
&= [-n(n-1)x^{n-2}+anx^{n-1}]Ce^{-ax}+[anx^{n-1}-a^2x^n]Ce^{-ax} \\
&= [-n(n-1)x^{n-2}+anx^{n-1}+anx^{n-1}-a^2x^n]Ce^{-ax} \\
&= [-n(n-1)x^{n-2}+2anx^{n-1}-a^2x^n]Ce^{-ax} \\
V''(x_0) &= \left[-n(n-1)\left(\frac{n}{a}\right)^{n-2}+2an\left(\frac{n}{a}\right)^{n-1}-a^2\left(\frac{n}{a}\right)^{n}\right]Ce^{-n} \\
&= \left[(-n^2+n)\left(\frac{n}{a}\right)^{n-2}+2an\left(\frac{n}{a}\right)^{n-1}-a^2\left(\frac{n}{a}\right)^{n}\right]Ce^{-n} \\
&= \left[-a^2\left(\frac{n}{a}\right)^{n}+a\left(\frac{n}{a}\right)^{n-1}+2an\left(\frac{n}{a}\right)^{n-1}-a^2\left(\frac{n}{a}\right)^{n}\right]Ce^{-n} \\
&= \left[-2a^2\left(\frac{n}{a}\right)^{n}+2an\left(\frac{n}{a}\right)^{n-1}+a\left(\frac{n}{a}\right)^{n-1}\right]Ce^{-n} \\
&= \left[-2a^2\left(\frac{n}{a}\right)^{n}+2a^2\left(\frac{n}{a}\right)^{n}+a\left(\frac{n}{a}\right)^{n-1}\right]Ce^{-n} \\
&= a\left(\frac{n}{a}\right)^{n-1}Ce^{-n}
\end{aligned}
$$

So that the frequency of small oscilations is

$$
\boxed{\omega=\sqrt{\frac{a}{m}\left(\frac{n}{a}\right)^{n-1}Ce^{-n}}}
$$

7. **Problem 7. Bead on a rotating hoop: A bead is free to slide along a frictionless hoop of radius R. The hoop rotates with constant angular speed ω around a vertical diameter (See Fig. 3). Find the equation of motion for the angle θ shown (angle is in radians). What are the equilibrium positions? What is the frequency of small oscillations about the stable equilibrium? There is one value of ω that is quite special. What is it and why is it special?**

   Solution: We will use the Lagrangian method to solve this problem. First, we must define our kinetic energy. We will do this by considering the simplest motions of the system. First, let us assume that $\omega = 0$ so that the bead can only move through the θ variable. From a consideration of polar coodinates it is clear that the kinetic energy contribution due to a change in θ is $\frac{1}{2}m(R\dot{\theta})$.

   Now, let θ be fixed at an arbitary angle $\theta = \theta_1$ and let us let $\omega \neq 0$. Then our former kinetic energy contribution disappears (because θ is constant) and we must find an alternate expression to account for the motion of the bead. If we let $\theta_1 = \pi/2$ then the bead points in a directions which is perpendicular to the spin axis and we may write the kinetic energy as $\frac{1}{2}mR^2\dot{\theta}^2$. Alternatively, if we let $\theta_1 = 0$ then the bead remains at the bottom of the hoop and has no kinetic energy[3]. Therefore we find that this part of the kinetic energy is defined by how far away (specifically the perpendicular distance) the bead is from the spin axis. In particular, the kinetic energy is defined by the perpendicular extent, $R\sin\theta$, of the bead from the spin axis. Allowing this perpendicular extent

[3]We are considering the bead as so small that any rotational motion about its center of mass contributes a neglible amount to its kinet energy.

to be our radius of rotational motion we find that this part of the kinetic energy is $\frac{1}{2}m(R\sin\theta_1\omega)^2$. So, in general, we find that the kinetic energy of the bead is

$$\text{KE} = \frac{1}{2}mR^2(\sin^2\theta\omega^2 + \dot{\theta}^2)$$

Now, we will find the potential energy. The bead is in a gravitational field so all of its potential energy must arise from the its height/vertical position. We can define this quantity with $R\cos\theta$. Also, the bead is constrained to move along the hoop so let us define zero potential energy to be at the bottom of the hoop $\theta = 0$ and maximum potential energy to be at the top of the hoop $\theta = \pi$. With these considerations we can write our potential energy function as[4]

$$\text{PE} = mgR(1 - \cos\theta)$$

We may now write our Lagraingian.

$$\mathcal{L} = \frac{1}{2}mR^2(\sin^2\theta\omega^2 + \dot{\theta}^2) - mgR(1 - \cos\theta)$$

Using the E-L equations to generate the EOM for θ we find

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\theta} &= mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta \\
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= \frac{d}{d}\left(mR^2\dot{\theta}\right) \\
&= mR^2\ddot{\theta}
\end{aligned}$$

So our EOM becomes

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} \quad \Longrightarrow \quad \boxed{\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta}$$

where we divided both sides of the E-L equation by mR^2 to simplify the result. In order, to find the equilibrium positions in this system, we must find the values of θ where $\ddot{\theta} = 0$. Setting the right hand side of our EOM to zero we find.

$$\begin{aligned}
0 &= \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta \\
&= \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)
\end{aligned}$$

The right hand side is zero if $\sin\theta = 0$ (i.e. if $\theta = 0, \pi$) or if the quantity in the parentheses is zero. So we find that our equilibrium positions occur at the angles

$$\boxed{\theta_1 = 0} \qquad \boxed{\theta_2 = \cos^{-1}\left(\frac{g}{R\omega^2}\right)} \qquad \boxed{\theta_3 = \pi}$$

To find the frequency of small oscillations we will apply Eq (2), but first we must define the **effective potential** of our system. Whenever we derive an EOM from the E-L equations we end up with a result which reproduces Newton's Law in a well defined potential

$$m\ddot{x} = -V'(x)$$

[4]The constant mgR is arbitary; $-mgR\cos\theta$ is the only essential part of the potential. We include the constant only to keep the potential energy exclusively positive.

And if we claim that the relevant domain of x exists near a **stable** equilibrium point x_0, then we may expand our potential function up to second order in x to find $V(x) \approx V(x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2$ and, dropping the constants, we find that our EOM can be written as

$$\ddot{x} + \frac{V''(x_0)}{m} x = 0$$

In this problem we are using angular variables but we should end up with a similar result. Namely, an equation of the form

$$\ddot{\theta} + \frac{V''(\theta_0)}{m}\theta = 0$$

near a stable equilibrium point θ_0. But, we have three equilibrium positions in this problem. In order to find which are stable we must compute $V''(\theta)$. From our standard EOM we may define $V(\theta)$ by the fact that the E-L equations imply

$$-V'(\theta) = m\left(\omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta\right)$$

So that

$$\begin{aligned} V''(\theta) &= -m\frac{d}{d\theta}(\omega^2 \sin\theta\cos\theta - \frac{g}{R}\sin\theta) \\ &= -m(\omega^2(\cos^2\theta - \sin^2\theta) - \frac{g}{R}\cos\theta) \end{aligned}$$

In order for an equilibrium point to be considered stable, the second derivative of the potential at that point must be positive (i.e. $V''(\theta_0) > 0$). Evaluating this second derivative at our first equlibrium point we find

$$\begin{aligned} V''(\theta_1 = 0) &= -m(\omega^2 - \frac{g}{R}) \\ &= m(\frac{g}{R} - \omega^2) \end{aligned}$$

So θ_1 is a stable equilibrium point if $\frac{g}{R} > \omega^2$. In this case, we find that the frequency of small oscilltions about the equilibrium point is

$$\boxed{\text{Frequency of Oscillations about } \theta_1 = \sqrt{\frac{1}{m}V''(\theta_1)} = \sqrt{\frac{g}{R} - \omega^2}}$$

. Now for the second equilibrium point.

$$\begin{aligned} V''(\theta_2 = \cos^{-1}\frac{g}{R\omega^2}) &= -m\left[\omega^2\left(\left(\frac{g}{R\omega^2}\right)^2 - \frac{(R\omega^2) - g^2}{(R\omega^2)^2}\right) - \frac{g}{R}\left(\frac{g}{R\omega^2}\right)\right] \\ &= -m\left[\frac{g^2 + g^2 - (R\omega^2)^2}{(R\omega)^2} - \frac{g^2}{(R\omega)^2}\right] \\ &= -m\left[\frac{g^2 - (R\omega^2)^2}{(R\omega)^2}\right] \\ &= m\left[\frac{(R\omega^2)^2 - g^2}{(R\omega)^2}\right] = m\left[\omega^2 - \frac{g^2}{(R\omega)^2}\right] \end{aligned}$$

So θ_2 is a stable equilibrium point if $\frac{g}{R} < \omega^2$. In this case, we find that the frequency of small oscillations about the equilibrium point is

$$\boxed{\text{Frequency of Oscillations about } \theta_2 = \sqrt{\frac{1}{m}V''(\theta_2)} = \sqrt{\omega^2 - \frac{g^2}{(R\omega)^2}}}$$

. Finally, for the last equilibrium point.

$$V''(\theta_3 = \pi) \quad = \quad -m(\omega^2 + \frac{g}{R})$$

Since this equation is always negative, θ_3 is never a stable equilibrium point. From our computation of the frequency of oscillations about the stable equilibria it is clear that the value of ω that is so "special" is $\boxed{\omega^2 = g/R}$. This value separates our two stable equilibrium cases and therfore defines whether the mass can oscillate near the bottom of the hoop ($\theta = 0$) or at some other point ($\theta = \cos^{-1} g/R\omega^2$)

8. **Problem 8. Moving Plane: A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ (in radians) [Figure 4]. The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane? Use Lagrangian to solve this (NOT $F = ma$). (It turns out you can use $F = ma$ to solve this problem, but this turns out to be a much more difficult way to solve this problem).**

   Solution: Contingent on how we define our coordinate systems we will obtain different Lagrangians, but all will result in the same value for the acceleration of the plane. Using the coordinate system depicted in the associated figure, we find that the block has a horizontal and vertical motion and that the plane only moves horizontally. From the fact that block is constrained to move along the plane it is clear that the x and y motions are not independent of each other and we therefore only need one of these components in our E-L equation. Allowing x_1 and y_1 to respectively be the absolute horizontal and vertical positions of the block we find that they can be written in terms of our system's coordinates (x, X)

$$\begin{aligned} asx_1 &= x + X \\ y_1 &= y = -x \tan\theta \end{aligned}$$

   The first line states that the absolute position of the block x_1 is the position x of the block relative to the wedge plus the absolute position X of the wedge. The second line states the same thing except the wedge has no changing vertical position which allows its Y component to be ignored. In the last equality we used the trigonometry of the right triangle in the figure write y in terms of x. The negative sign must be included because as the block slides farther down the plane, its horizontal position x increases but its vertical position y decreases. With this information we can find the kinetic energy of our system

$$\text{KE} = \frac{1}{2}m((\dot{x} + \dot{X})^2 + \dot{x}^2 \tan^2\theta) + \frac{1}{2}M\dot{X}^2$$

   the first term represents the kinetic energy due to the two-dimensional motion of the block, and the second term represents to kinetic energy due to the one-dimensional motion of the wedge. We can also find the potential energy of our system.

$$\text{PE} = -mgx \tan\theta$$

with this information we can find the Lagrangian of our system and then the EOMs. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m((\dot{x}+\dot{X})^2 + \dot{x}^2\tan^2\theta) + \frac{1}{2}M\dot{X}^2 + mgx\tan\theta$$

and by applying the E-L equations to the independent coordinates x and X, we have

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} \implies \boxed{m(\ddot{x}+\ddot{X}) + m\ddot{x}\tan^2\theta = mg\tan\theta}$$

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}} - \frac{\partial\mathcal{L}}{\partial X} \implies \boxed{m(\ddot{x}+\ddot{X}) + M\ddot{X} = 0}$$

These two EOMs provide us with a system of two equations in $\ddot{x}$ and $\ddot{X}$ which we can solve by elimination. From the second equation we find

$$\ddot{x} = -\frac{m+M}{m}\ddot{X}$$

which, when plugged into the second equation, gives

$$\begin{aligned}
mg\tan\theta &= m(\ddot{x}+\ddot{X}) + m\ddot{x}\tan^2\theta \\
&= m\ddot{x}(1+\tan^2\theta) + m\ddot{X} \\
&= \ddot{x}\sec^2\theta + \ddot{X} \\
&= -m\frac{m+M}{m}\ddot{X}\sec^2\theta + m\ddot{X} \\
&= \ddot{X}(m - (m+M)\sec^2\theta) \\
&= \ddot{X}(m(1-\sec^2\theta) - M\sec^2\theta) \\
&= -\ddot{X}(m\tan^2\theta + M\sec^2\theta) \\
\ddot{X} &= -\frac{mg\tan\theta}{m\tan^2\theta + M\sec^2\theta} \implies \boxed{\ddot{X} = -\frac{mg\sin\theta\cos\theta}{m\sin^2\theta + M}}
\end{aligned}$$

As is customary, it is always a good idea to check **units** and **limiting cases**. We have trig functions which have no units so they need not be considered. We then only have an mg in the numerator and a mass term in the denominator. Cancelling the units of mass from the equation we find that $\ddot{X}$ has units of g which has units of acceleration. So the units are correct. Now to check limiting cases. If $M >> m$ we expect the acceleration of M to go to zero because the m is too small to push M. This result is also predicted by our equation as seen when we divide the numerator and denominator by M and take the limit as $m/M \to 0$. The opposite case ($m >> M$) is difficult to intuitively predict but it is clear that $\ddot{X} \neq 0$ which is what we get in the limit as $m/M \to \infty$.