MITES 2010: Physics III
Survey of Modern Physics
Problem Set 4 Solutions

1. **Problem 1: Applying the rules of Quantum Mechanics.**
Consider a particle whose energy can only take on the following values:

$E_1 = 3\hbar\omega, \;\; E_2 = 6\hbar\omega, \;\; E_3 = 9\hbar\omega, \;\; E_4 = 0$

These are the only allowed energies that the particle can have. They have corresponding eigenstates:
$|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$, $|E_4\rangle$. $\hat{H}$ **is the Hamiltonian operator - operator representing measurement of the particle's energy.**

Consider a particle in the following indeterminate state.
$|\psi\rangle = \frac{-1}{\sqrt{6}}|E_1\rangle + \frac{e^{i\theta}}{\sqrt{6}}|E_2\rangle + \sqrt{\frac{2}{3}}e^{i\varphi}|E_4\rangle$

(i.) **If you measure the energy of 10,000 identical particles, all in the above 'indeterminate' state just before your measurement, how many of them do you expect to yield energy value of E_2?**

Solution: The probability of obtaining E_2 is $\left|\frac{e^{i\theta}}{\sqrt{6}}\right|^2 = \frac{1}{6}$. So, out of 10,000 measurements, we expect

$$\frac{10,000}{6} \approx \boxed{1667 \text{ particles}}$$

to yield energy E_2.

(ii.) **On average, what value of energy would your measurement yield?**

Solution:

$$\begin{aligned} E_{avg} &= |c_1|^2 E_1 + |c_2|^2 E_2 + |c_3|^2 E_3 + |c_4|^2 E_4 \\ &= \frac{1}{6} 3\hbar\omega + \frac{1}{6} 6\hbar\omega + 0 \cdot 9\hbar\omega + \frac{2}{3} \cdot 0 \\ &= \boxed{\frac{3}{2}\hbar\omega} \end{aligned}$$

(iii.) **For which value of θ is the probability of a particle having energy E_2 zero?**

Solution: $\boxed{\text{There is no value of } \theta \text{ for which the probability of obtaining } E_2 \text{ is zero}}$. First of all, this is because the coefficient of the state $|E_2\rangle$ is proportional to $e^{i\theta}$ and $e^{i\theta}$ never equals zero. Second of all, the magnitude of $e^{i\theta}$ is always 1 regardless of θ, so the probability of being in E_2 is independent of θ Prob(E_2)=$|e^{i\theta}/\sqrt{6}|$.

(iv.) **Suppose you measure the energy of a particle. You find that it has energy E_1. Does this mean that the particle was in state $|E_1\rangle$ just before your measurement? Suppose you measure the energy of the same particle for the second time. What is the probability that you get E_4 as the energy this second time?**

Solution: $\boxed{\text{No, the particle was not in the state } |E_1\rangle \text{ before the measurement}}$. A postulate of quantum mechanics states that a quantum state is in a superposition of many states before a measurement and only collapses to one state after the measurement. Also, since the state is now in $|E_1\rangle$ then there is $\boxed{\text{zero probability that it is in any other state}}$.

(v.) **Suppose you measure the energy of the particle and find that it has energy E_2. Write down the ket-vector representing the particle's state immediately after this measurement.**

Solution: Since we measured energy and obtained the value E_2 we know that the particle's state has now collapsed to the state corresponding to E_2. That is, the particle is in state $\boxed{|\psi\rangle = |E_2\rangle}$.

2. **Problem 2. Sequential measurements**
An operator $\hat{A}$ representing observable A, has two normalized eigenstates ψ_1 and ψ_2, with eigenvalues a_1 and a_2 respectively. Operator $\hat{A}$ representing observable B, has two normalized eigenstates ϕ_1 and ϕ_2, with eigenvalues b_1 and b_2. The eigenstates are related by

$$\psi_1 = \frac{3\phi_1 + 4\phi_2}{5} \qquad \psi_2 = \frac{4\phi_1 - 3\phi_2}{5} \tag{1}$$

(a.) **Observable A is measured, and the value a_1 is obtained. What is the state of the system (immediately) after this measurement?**

Solution: By one of the postulates of quantum mechanics, the state has collapsed into the state corresponding to eigenvalue a_1. So, $\boxed{\text{the state is in } \psi_1}$.

(b.) **If B is now measured, what are the possible results, and what are their probabilities?**

Solution: The state is currently in ψ_1.

$$\psi_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$$

From this state, there are two possible results upon measurement of B. These results and their associated probabilities are

Results	Probability
b_1	$\left(\frac{3}{5}\right)^2 = \frac{9}{25}$
b_2	$\left(\frac{4}{5}\right)^2 = \frac{16}{25}$

which fits our normalization requirement because $9/25 + 16/25 = 25/25 = 1$

(c.)**Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note that the answer would be quite different if I had told you the outcome of the B measurement.)**

Solution: There are two possible ways to get to a_1 from our initial a_1 measurement. We can obtain b_1 from the B measurement and then obtain a_1 from the A measurement. Or, we can obtain b_2 from the B measurement and then obtain a_1 from the A measurement. In order to calculate the total probability of obtaining a_1, we must calculate the probability of these independent paths and then add them. But first, we must write ϕ_1 and ϕ_2 in terms of ψ_1 and ψ_2 in order to obtain the probabilities for the second measurement of A after our measurement of B. Solving Eq (1)by elimination, we find for ϕ_1

$$\begin{aligned} \psi_1 &= \frac{3\phi_1 + 4\phi_2}{5} \\ \frac{5\psi_1 - 4\phi_2}{3} &= \phi_1 \end{aligned}$$

Substituting this result into our other equation, we find

$$\begin{aligned} \psi_2 &= \frac{4\phi_1 - 3\phi_2}{5} \\ 5\psi_2 &= \frac{4}{3}(5\psi_1 - 4\phi_2) - 3\phi_2 \\ &= \frac{20}{3}\psi_1 - \frac{16}{3}\phi_2 - 3\phi_2 \\ &= \frac{20}{3}\psi_1 - \frac{25}{3}\phi_1 \\ 25\phi_2 &= \frac{20}{3}\psi_1 - 5\psi_2 \\ \phi_2 &= \frac{4\psi_1 - 3\psi_2}{5} \end{aligned}$$

Now, solving for ϕ_1

$$\begin{aligned} 5\psi_1 &= 3\phi_1 + 4(\frac{4}{5}\psi_1 - \frac{3}{5}\psi_2) \\ &= 3\phi_1 + \frac{16}{5}\psi_1 - \frac{12}{5}\psi_2 \\ \frac{9}{5}\psi_1 &= 3\phi_1 - \frac{12}{5}\psi_2 \\ \phi_1 &= \frac{3\psi_1 + 4\psi_2}{5} \end{aligned}$$

So, in summary, we have

$$\phi_1 = \frac{3\psi_1 + 4\psi_2}{5} \qquad \phi_2 = \frac{4\psi_1 - 3\psi_2}{5}$$

From this result, we may now construct our probabilities. There are two paths to get to a_1 from a measurement of B. These paths and their associated probabilities are

Path	Probability
1) From b_1 to a_1	$\frac{9}{25} \times \frac{9}{25} = \frac{81}{625}$
2) From b_2 to a_1	$\frac{16}{25} \times \frac{16}{25} = \frac{256}{625}$

For each path, we multiplied the probability of obtaining our first measured value of B by the probability of obtaining our second measured value of A. The total probability of obtaining a_1 is the sum of these probabilities and is therefore $\frac{81}{625} + \frac{256}{625} = \boxed{\frac{337}{625}}$ We can check that this procedure for calculating probability is consistent with normalization by calculating the probability other two possible paths. Instead of obtaining a_1, we could have obtained a_2 in our second measurement and the paths and probabilities associated with this value are

Path	Probability
3) From b_1 to a_2	$\frac{9}{25} \times \frac{16}{25} = \frac{144}{625}$
4) From b_2 to a_2	$\frac{16}{25} \times \frac{9}{25} = \frac{144}{625}$

We have listed all possible outcomes of our measurements. The sum of the probabilities for theses outcomes must be one.

$$\text{Prob(Path 1)} + \text{Prob(Path 2)} + \text{Prob(Path 3)} + \text{Prob(Path 4)} = \frac{81}{625} + \frac{256}{625} + \frac{144}{625} + \frac{144}{625} = \frac{625}{625} = 1$$

Our outcomes are normalized, so we know that our procedure is correct.

3. **Problem 3: Three Level System.**
The Hamiltonian for a certain three-level system is represented by the matrix

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Two other observables, A and B, are represented by the matrices

$$\hat{A} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \hat{B} = \mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(a.) **Find the eigenvalues and (normalized) eigenvectors of H, A, and B.**

Solution: Employing the standard procedure of calculating eigenvalues/eigenvectors, we find

$$\begin{aligned} \det(\hat{H} - EI) &= \det \begin{pmatrix} \hbar\omega - E & 0 & 0 \\ 0 & 2\hbar\omega - E & 0 \\ 0 & 0 & 2\hbar\omega - E \end{pmatrix} \\ &= (\hbar\omega - E)\det \begin{pmatrix} 2\hbar\omega - E & 0 \\ 0 & 2\hbar\omega - E \end{pmatrix} \\ &= (\hbar\omega - E)(2\hbar\omega - E)^2 \end{aligned}$$

So the possible eigenvalues of $\hat{H}$ are $E_1 = \hbar\omega$, $E_2 = 2\hbar\omega$, and $E_3 = 2\hbar\omega$. Using these eigenvalues to find the eigenvectors, we find

$\underline{E_1 = \hbar\omega}$

$$
\begin{aligned}
0 = (\hat{H} - E_1 I)|E_1\rangle &= \begin{pmatrix} \hbar\omega - \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega - \hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega - \hbar\omega \end{pmatrix} |E_1\rangle \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar\omega & 0 \\ 0 & 0 & \hbar\omega \end{pmatrix} |E_1\rangle
\end{aligned}
$$

So we see that $|E_1\rangle$ can have no second and third component in order for it to satisfy the above equation. Therefore, $|E_1\rangle$ must be of the form

$$
|E_1\rangle = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
$$

Where we chose $a = 1$ in order to normalize $|E_1\rangle$.

The next eigenvalue $E_2 = E_3 = 2\hbar\omega$ has multiplicity two (i.e. it occurs twice) so we should get two eigenstates.

$\underline{E = 2\hbar\omega}$

$$
\begin{aligned}
0 = (\hat{H} - EI)|E\rangle &= \begin{pmatrix} \hbar\omega - 2\hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega - 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega - 2\hbar\omega \end{pmatrix} |E\rangle \\
&= \begin{pmatrix} -\hbar\omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} |E\rangle
\end{aligned}
$$

We see that $|E\rangle$ can have no first component in order for it to satisfy the above equation. There are two ways we can accomplish this: $|E\rangle$ only has a second component; $|E\rangle$ only has a third component. These two ways correspond to our two eigenstates, so that we have $|E_2\rangle$ and $|E_3\rangle$ as

$$
|E_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad\qquad |E_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Summarizing these results, we have for the operator $\hat{H}$

Eigenvalues	Eigenstates
$E_1 = \hbar\omega$	$\|E_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
$E_2 = 2\hbar\omega$	$\|E_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$E_3 = 2\hbar\omega$	$\|E_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Employing the same procedure for $\hat{A}$, we find

$$\begin{aligned}
\det(\hat{A} - aI) &= \det \begin{pmatrix} -a & \mu & 0 \\ \mu & -a & 0 \\ 0 & 0 & 2\mu - a \end{pmatrix} \\
&= -a \det \begin{pmatrix} -a & 0 \\ 0 & 2\mu - a \end{pmatrix} - \mu \begin{pmatrix} \mu & 0 \\ 0 & 2\mu - a \end{pmatrix} \\
&= -a(a - 2\mu)a - \mu\mu(2\mu - a) \\
&= a^2(2\mu - a) - \mu^2(2\mu - a) \\
&= (a^2 - \mu^2)(2\mu - a) \\
&= (a - \mu)(a + \mu)(2\mu - a)
\end{aligned}$$

So, the eigenvalues of the operator $\hat{A}$ are $a_1 = \mu$, $a_2 = -\mu$, and $a_3 = 2\mu$. Computing eigenvectors, we find

$\underline{a_1 = \mu}$

$$\begin{aligned}
0 = (\hat{A} - a_1 I)|A_1\rangle &= \begin{pmatrix} -\mu & \mu & 0 \\ \mu & -\mu & 0 \\ 0 & 0 & 2\mu - \mu \end{pmatrix} |A_1\rangle \\
&= \begin{pmatrix} -\mu & \mu & 0 \\ \mu & -\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} |A_1\rangle
\end{aligned}$$

The last line leads us to an $|A_1\rangle$ of the form

$$|A_1\rangle = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

where we chose $\alpha_1 = 1/\sqrt{2}$ for normalization.

$\underline{a_2 = -\mu}$

$$\begin{aligned}
0 = (\hat{A} - a_2 I)|A_2\rangle &= \begin{pmatrix} \mu & \mu & 0 \\ \mu & \mu & 0 \\ 0 & 0 & 2\mu + \mu \end{pmatrix} |A_2\rangle \\
&= \begin{pmatrix} \mu & \mu & 0 \\ \mu & \mu & 0 \\ 0 & 0 & 3\mu \end{pmatrix} |A_2\rangle
\end{aligned}$$

The last line leads us to an $|A_2\rangle$ of the form

$$|A_1\rangle = \alpha_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

where we chose $\alpha_2 = 1/\sqrt{2}$ for normalization.

Lastly $\underline{a_3 = 2\mu}$

$$
\begin{aligned}
0 = (\hat{A} - a_3 I)|A_1\rangle &= \begin{pmatrix} -2\mu & \mu & 0 \\ \mu & -2\mu & 0 \\ 0 & 0 & 2\mu - 2\mu \end{pmatrix} |A_3\rangle \\
&= \begin{pmatrix} -2\mu & \mu & 0 \\ \mu & -2\mu & 0 \\ 0 & 0 & 0 \end{pmatrix} |A_3\rangle
\end{aligned}
$$

If we let $|A_3\rangle$ have the form

$$|A_3\rangle = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

then the last line suggests the following system of equations

$$
\begin{aligned}
-2g_1 + g_2 &= 0 \\
g_1 - 2g_2 &= 0
\end{aligned}
$$

these two equations are inconsistent and therefore produce the null solution $g_1 = g_2 = 0$. Therefore, $|A_3\rangle$ is of the form

$$|A_3\rangle = \begin{pmatrix} 0 \\ 0 \\ g_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where we chose $g_3 = 1$ for normalization. In summary, we have for the operator $\hat{A}$

Eigenvalues	Eigenstates
$a_1 = \mu$	$\|A_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
$a_2 = -\mu$	$\|A_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$
$a_3 = 2\mu$	$\|A_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Now, for $\hat{B}$. Studying the matrix form of this operator and comparing it with the operator $\hat{A}$, we see that $\hat{B}$ and $\hat{A}$ have the same set of eigenvalues and eigenvectors. We will prove this by explicit computation

$$
\begin{aligned}
\det(\hat{B} - bI) &= \det \begin{pmatrix} 2\lambda - b & 0 & 0 \\ 0 & -b & \lambda \\ 0 & \lambda & -b \end{pmatrix} \\
&= (2\lambda - b) \det \begin{pmatrix} -b & \lambda \\ \lambda & -b \end{pmatrix} \\
&= (2\lambda - b)(b^2 - \lambda^2) \\
&= (2\lambda - b)(b + \lambda)(b - \lambda)
\end{aligned}
$$

So the eigenvalues of the operator $\hat{B}$ are $b_1 = \lambda$, $b_2 = -\lambda$, and $b_3 = 2\lambda$. Computing eigenvectors, we find

$\underline{b_1 = \lambda}$

$$\begin{aligned} 0 = (\hat{B} - b_1 I)|B_1\rangle &= \begin{pmatrix} 2\lambda - \lambda & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & \lambda & -\lambda \end{pmatrix} |B_1\rangle \\ &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & \lambda \\ 0 & \lambda & -\lambda \end{pmatrix} |B_1\rangle \end{aligned}$$

This result is similar to one obtained for the $\hat{A}$ case and we therefore see that

$$|B_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{b_2 = -\lambda}$

$$\begin{aligned} 0 = (\hat{B} - b_2 I)|B_1\rangle &= \begin{pmatrix} 2\lambda + \lambda & 0 & 0 \\ 0 & +\lambda & \lambda \\ 0 & \lambda & +\lambda \end{pmatrix} |B_1\rangle \\ &= \begin{pmatrix} 3\lambda & 0 & 0 \\ 0 & \lambda & \lambda \\ 0 & \lambda & \lambda \end{pmatrix} |B_1\rangle \end{aligned}$$

This result is similar to one obtained for the $\hat{A}$ case and we therefore see that

$$|B_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\underline{b_3 = 2\lambda}$

$$\begin{aligned} 0 = (\hat{B} - b_3 I)|B_1\rangle &= \begin{pmatrix} 2\lambda 2\lambda & 0 & 0 \\ 0 & -2\lambda & \lambda \\ 0 & \lambda & -2\lambda \end{pmatrix} |B_1\rangle \\ &= \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -2\lambda & \lambda \\ 0 & \lambda & -2\lambda \end{pmatrix} |B_1\rangle \end{aligned}$$

Once again, we obtain inconsistent equations from two of our rows (the bottom two rows) and the only surviving component is the top component. This result is similar to one obtained for the $\hat{A}$ case and we therefore see that

$$|B_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In summary, we have for the operator $\hat{B}$

Eigenvalues	Eigenstates
$b_1 = \lambda$	$\|B_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
$b_2 = -\lambda$	$\|B_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$
$b_3 = 2\lambda$	$\|B_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(b.) **Suppose the system starts out in the generic state**

$$|S(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$. **Find the expectation values (i.e. averages) (at $t = 0$) of H, A and B.**

Solution: The formula for the average of an observable

$$\langle O \rangle = \langle S(0)|\hat{O}|\hat{S}(0)\rangle$$

So for the Hamiltonian $|hatH$

$$\begin{aligned} \langle S(0)|\hat{H}|\hat{S}(0)\rangle &= \begin{pmatrix} c_1^\dagger & c_2^\dagger & c_3^\dagger \end{pmatrix} \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} c_1^\dagger & c_2^\dagger & c_3^\dagger \end{pmatrix} \begin{pmatrix} c_1\hbar\omega \\ 2c_2\hbar\omega \\ 2c_3\hbar\omega \end{pmatrix} = \boxed{\hbar\omega(|c_1|^2 + 2|c_2|^2 + 2|c_3|)^2 = E_{avg}} \end{aligned}$$

similarly for $\hat{A}$.

$$\begin{aligned} \langle S(0)|\hat{A}|\hat{S}(0)\rangle &= \begin{pmatrix} c_1^\dagger & c_2^\dagger & c_3^\dagger \end{pmatrix} \lambda \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\ &= \begin{pmatrix} c_1^\dagger & c_2^\dagger & c_3^\dagger \end{pmatrix} \begin{pmatrix} \lambda c_2 \\ \lambda c_1 \\ 2\lambda c_3 \end{pmatrix} = \boxed{\lambda(c_1^\dagger c_2 + c_2^\dagger c_1 + 2|c_3|^2) = a_{avg}} \end{aligned}$$

And finally for $\hat{B}$

$$
\begin{aligned}
\langle S(0)|\hat{A}|\hat{S}(0)\rangle &= \begin{pmatrix} c_1^\dagger & c_2^\dagger & c_3^\dagger \end{pmatrix} \lambda \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \begin{pmatrix} c_1^\dagger & c_2^\dagger & c_3^\dagger \end{pmatrix} \begin{pmatrix} 2\mu c_1 \\ \mu c_3 \\ \mu c_2 \end{pmatrix} = \boxed{\mu(|c_1|^2 + c_2^\dagger c_3 + c_3^\dagger c_2) = b_{avg}}
\end{aligned}
$$

(c.) **What is $|S(t)\rangle$? If you measured the energy of this state (at time t), what values might you get, and what is the probability of each? Answer the same questions for A and for B.**

Solution: From the notes we know that if $|S(0)\rangle$ is of the form

$$|S(0)\rangle = a|E_1\rangle + b|E_2\rangle$$

then the time dependent state $|S(t)\rangle$ is

$$|S(t)\rangle = ae^{-iE_1/\hbar}|E_1\rangle + be^{-iE_2/\hbar}|E_2\rangle$$

So we see that our $|S(t)\rangle$ is of the form

$$|S(t)\rangle = \begin{pmatrix} c_1 e^{-iE_1/\hbar} \\ c_2 e^{-iE_2/\hbar} \\ c_3 e^{-iE_3/\hbar} \end{pmatrix} = \boxed{\begin{pmatrix} c_1 e^{-i\omega} \\ c_2 e^{-i2\omega} \\ c_3 e^{-i2\omega} \end{pmatrix}}$$

and from the form of $|S(0)\rangle$ as

$$|S(0)\rangle = c_1|E_1\rangle + c_2|E_2\rangle + c_3|E_3\rangle,$$

we see that if we measure the energy of our state, then we can get

Energy	Probability
$\hbar\omega$	$\|c_1\|^2$
$2\hbar\omega$	$\|c_2\|^2 + \|c_3\|^2$

where we added the probabilities of being in state $|E_2\rangle$ and $|E_3\rangle$ because they have the same eigenvalue.

In order to find the probabilities of obtaining certain eigenvalues of the operators $\hat{A}$ and $\hat{B}$, it is useful to write $|S(0)\rangle$ as a linear combination of their respective eigenvectors. For $\hat{A}$ we have

$$\begin{aligned}
|S(0)\rangle &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \begin{pmatrix} (c_1 + c_2 + c_1 - c_2)/2 \\ (c_1 + c_2 - c_1 + c_2)/2 \\ c_3 \end{pmatrix} \\
&= c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{c_1 + c_2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{c_1 - c_2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{c_1 + c_2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{c_1 - c_2}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
&= c_3 |A_3\rangle + \frac{c_1 + c_2}{\sqrt{2}} |A_1\rangle + \frac{c_1 - c_2}{\sqrt{2}} |A_2\rangle
\end{aligned}$$

So, we see that if we measure observable A, then we get

Eigenvalue	Probability
$a_1 = \mu$	$\frac{\lvert c_1 + c_2\rvert^2}{2}$
$a_2 = -\mu$	$\frac{\lvert c_1 - c_2\rvert^2}{2}$
$a_3 = 2\mu$	$\lvert c_3\rvert^2$

Similarly decomposing $|S(0)\rangle$ into the eigenstates of $\hat{B}$, we find

$$\begin{aligned}
|S(0)\rangle &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \\
&= \begin{pmatrix} c_1 \\ (c_2 + c_3 + c_2 - c_3)/2 \\ (c_2 + c_3 - c_2 + c_3)/2 \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{c_2 + c_3}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{c_2 - c_3}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
&= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{c_2 + c_3}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{c_2 - c_3}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
&= c_1 |B_3\rangle + \frac{c_2 + c_3}{\sqrt{2}} |B_1\rangle + \frac{c_2 - c_3}{\sqrt{2}} |B_2\rangle
\end{aligned}$$

So, we see that if we measure observable B, then we get

Eigenvalue	Probability
$b_1 = \lambda$	$\frac{\|c_2 + c_3\|^2}{2}$
$b_2 = -\lambda$	$\frac{\|c_2 - c_3\|^2}{2}$
$b_3 = 2\lambda$	$\|c_1\|^2$

4. **Problem 4. Some commutator relations.**
(a.) **Prove the following commutator identity:**

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$$

Solution: Beginning with the right hand side

$$\begin{aligned} \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} &= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + (\hat{A}\hat{C} - \hat{C}\hat{A})\hat{B} \\ &= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{A}\hat{C}\hat{B} - \hat{C}\hat{A}\hat{B} \\ &= \hat{A}\hat{B}\hat{C} - \hat{C}\hat{A}\hat{B} \\ &= [\hat{A}\hat{B}, \hat{C}] \end{aligned}$$

So, the equality is proven. (b.) **Show that**

$$[x^n, \hat{p}] = i\hbar n x^{n-1}$$

Solution: As usual we include a dummy function to absorb our differentiation

$$\begin{aligned} [x^n, \hat{p}]f(x) = \left[x^n, \frac{\hbar}{i}\frac{d}{dx}\right]f(x) &= x^n \frac{\hbar}{i}\frac{d}{dx}f(x) - \frac{\hbar}{i}\frac{d}{dx}(x^n f(x)) \\ &= x^n \frac{\hbar}{i}\frac{d}{dx}f(x) - x^n \frac{\hbar}{i}\frac{d}{dx}f(x) - \frac{h}{i} n x^{n-1} f(x) \\ &= -\frac{\hbar}{i} n x^{n-1} f(x) \quad \Longrightarrow \quad \boxed{[x^n, \hat{p}] = i\hbar n x^{n-1}} \end{aligned}$$

(c.) **Show more generally that**

$$[f(x), \hat{p}] = i\hbar \frac{df}{dx}$$

for any function $f(x)$.

Solution: Once again employing the dummy function procedure

$$\begin{aligned} [f(x), \hat{p}]g(x) = \left[f(x), \frac{\hbar}{i}\frac{d}{dx}\right]g(x) &= f(x)\frac{\hbar}{i}\frac{d}{dx}g(x) - \frac{\hbar}{i}\frac{d}{dx}(g(x)f(x)) \\ &= f(x)\frac{\hbar}{i}\frac{d}{dx}g(x) - f(x)\frac{\hbar}{i}\frac{d}{dx}g(x) - g(x)\frac{h}{i}\frac{d}{dx}f(x) \\ &= -g(x)\frac{h}{i}\frac{d}{dx}f(x) \quad \Longrightarrow \quad \boxed{[f(x), \hat{p}] = i\hbar \frac{df}{dx}} \end{aligned}$$

5. **Problem 5. Quantum Simple Harmonic Oscillator**
In this problem, we derive the properties of the simple harmonic oscillator that exists in the realm of quantum mechanics.
The Energy (Hamiltonian operator) of a simple harmonic oscillator with mass M and angular frequency ω in classical mechanics is

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

Quantizing above classical Hamiltonian yields the quantum mechanical Hamiltonian for a simple harmonic oscillator which is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2$$

At a first glance, this looks just like the classical Hamiltonian. But don't be fooled - the important difference here is that the momentum and position are represented by the operators $\hat{p}$ and $\hat{x}$ respectively, since both are observables. (Remember that in quantum mechanics any observable (things you can measure) are represented by an operator).

(a.) Let us define the following non-Hermitian operator $\hat{a}$ and its Hermitian conjugate $\hat{a}^\dagger$:

$$a = \frac{\omega m\hat{x} + i\hat{p}}{\sqrt{2\omega m\hbar}} \qquad a^\dagger = \frac{\omega m\hat{x} - i\hat{p}}{\sqrt{2\omega m\hbar}}$$

Inverting these relations, derive the following position and momentum operators $\hat{x}$ and $\hat{p}$:

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \\ \hat{p} &= -i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a} - \hat{a}^\dagger) \end{aligned}$$

Solution: Adding $\hat{a}$ and $\hat{a}^\dagger$ we find

$$\begin{aligned} \hat{a} + \hat{a}^\dagger &= \frac{\omega m\hat{x} + i\hat{p}}{\sqrt{2\omega m\hbar}} + \frac{\omega m\hat{x} - i\hat{p}}{\sqrt{2\omega m\hbar}} \\ &= \frac{2\omega m\hat{x}}{\sqrt{2\omega m\hbar}} \\ &= \sqrt{\frac{2m\omega}{\hbar}}\hat{x} \end{aligned}$$

so that $\boxed{\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)}$

Similarly, we could subtract $\hat{a}^\dagger$ from $\hat{a}$ to obtain

$$\begin{aligned} \hat{a} - \hat{a}^\dagger &= \frac{\omega m\hat{x} + i\hat{p}}{\sqrt{2\omega m\hbar}} - \frac{\omega m\hat{x} - i\hat{p}}{\sqrt{2\omega m\hbar}} \\ &= \frac{2i\hat{p}}{\sqrt{2\omega m\hbar}} \\ &= i\sqrt{\frac{2}{\omega m\hbar}}\hat{p} \end{aligned}$$

so that $\boxed{\hat{p} = -i\sqrt{\frac{\omega m\hbar}{2}}(\hat{a} - \hat{a}^\dagger)}$

(b.) **Using the commutation relation** $[\hat{x}, \hat{p}]$ **that we derived in class, show the following commutation relation:**

$$[\hat{a}, \hat{a}^\dagger] = 1$$

Solution: We calculate the commutator $[\hat{x}, \hat{p}]$ with our position and momentum operators expressed in terms of $\hat{a}$ and $\hat{a}^\dagger$.

$$\begin{aligned}
[\hat{x}, \hat{p}] &= -i\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\omega m\hbar}{2}}(\hat{a} + \hat{a}^\dagger)(\hat{a} - \hat{a}^\dagger) \\
& \quad i\sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\omega m\hbar}{2}}(\hat{a} - \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) \\
&= -i\frac{\hbar}{2}(\hat{a}^2 + \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a^2}^\dagger) \\
& \quad +i\frac{\hbar}{2}(\hat{a}^2 - \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a^2}^\dagger) \\
&= i\hbar(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}) \\
&= i\hbar[\hat{a}, \hat{a}^\dagger]
\end{aligned}$$

And since $[\hat{x}, \hat{p}] = i\hbar$ we have $\boxed{[\hat{a}, \hat{a}^\dagger] = 1}$

(c.) **Let us define a characteristic length of an oscillator** $x_0 = \sqrt{\frac{\hbar}{m\omega}}$**. Then show that**

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + x_0\frac{d}{dx}\right) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} - x_0\frac{d}{dx}\right)
\end{aligned}$$

Solution: Simplifying $\hat{a}$, we have

$$\begin{aligned}
\hat{a} &= \frac{\omega m\hat{x} + i\hat{p}}{\sqrt{2\omega m\hbar}} \\
&= \frac{\omega m}{\sqrt{2\omega m\hbar}}\hat{x} + \frac{i}{\sqrt{2\omega m\hbar}}\frac{\hbar}{i}\frac{d}{dx} \\
&= \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega m}{\hbar}}\hat{x} + \sqrt{\frac{\hbar}{\omega m}}\frac{d}{dx}\right) \\
&= \boxed{\frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} + x_0\frac{d}{dx}\right)}
\end{aligned}$$

The derivation for $\hat{a}^\dagger$ is exactly the same as the one for $\hat{a}$ except we have a minus, instead of a plus, sign infront of the momentum operator. Therefore for $\hat{a}^\dagger$ we have

$$\boxed{\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{x_0} - x_0\frac{d}{dx}\right)}$$

(d.) **Now, putting everything together, show that the (quantum) Hamiltonian for a simple harmonic oscillator is**

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

So we have now reduced the problem to that of finding the eigenvalues of the occupation number operator

$$\hat{n} = \hat{a}^\dagger\hat{a}$$

Solution: Writing out our Hamiltonian Operator, we have

$$\begin{aligned}
\hat{H} &= \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \\
&= -\frac{\hbar m\omega}{2}\frac{(\hat{a}-\hat{a}^\dagger)^2}{2m} + \frac{m\omega^2}{2}\frac{\hbar}{2m\omega}(\hat{a}+\hat{a}^\dagger)^2 \\
&= -\frac{\hbar\omega}{4}(\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a^2}^\dagger) \\
&\quad +\frac{\hbar\omega}{4}(\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a^2}^\dagger) \\
&= \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\
&= \frac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + 1 + \hat{a}^\dagger\hat{a}) \\
&= \frac{\hbar\omega}{2}(2\hat{a}^\dagger\hat{a} + 1) = \boxed{\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}
\end{aligned}$$

In the second to last line, we used the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ to write $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$.

(e.) **Using the result from problem 4a, show that**

$$\begin{aligned}
[\hat{n}, \hat{a}^\dagger] &= \hat{a}^\dagger \\
[\hat{n}, \hat{a}] &= -\hat{a}
\end{aligned}$$

Solution: The result from problem 4a is

$$[AB, C] = A[B, C] + [A, C]B$$

Using this result in the following commutation relation, we have

$$\begin{aligned}
[\hat{n}, \hat{a}^\dagger] &= [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] \\
&= \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} \\
&= \hat{a}^\dagger \cdot 1 + 0 \\
&= \hat{a}^\dagger
\end{aligned}$$

Similarly, for the other commutation relation

$$\begin{aligned}[\hat{n}, \hat{a}] &= [\hat{a}^\dagger \hat{a}, \hat{a}] \\ &= \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} \\ &= 0 + (-1)\hat{a} \\ &= -\hat{a}\end{aligned}$$

In both derivations, we used the following results:

$$\begin{aligned}[\hat{a}, \hat{a}^\dagger] &= 1 \\ [\hat{a}, \hat{a}] &= [\hat{a}^\dagger, \hat{a}^\dagger] = 0\end{aligned}$$

The first result was derived in (b). The second result is the obvious statement that $\hat{a}\hat{a} = \hat{a}\hat{a}$ and $\hat{a}^\dagger \hat{a}^\dagger = \hat{a}^\dagger \hat{a}^\dagger$. (f.) **Now, show that if $|n\rangle$ is an eigenstate of $\hat{n}$ with an eigenvalue n, then $\hat{a}|n\rangle$ is an eigenstate of $\hat{n}$ with an eigenvalue $n + 1$.**

Solution: As a given, we have

$$\hat{n}|n\rangle = \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle \tag{2}$$

Calculating $\hat{n}\hat{a}^\dagger|n\rangle$, we find

$$\begin{aligned}\hat{n}\hat{a}^\dagger|n\rangle &= \hat{a}^\dagger \hat{a}\hat{a}^\dagger|n\rangle \\ &= \hat{a}^\dagger(\hat{a}^\dagger \hat{a} + 1)|n\rangle \\ &= \hat{a}^\dagger \hat{a}^\dagger \hat{a}|n\rangle + \hat{a}^\dagger|n\rangle \\ &= \hat{a}^\dagger n|n\rangle + \hat{a}^\dagger|n\rangle \\ &= (n+1)\hat{a}^\dagger|n\rangle\end{aligned}$$

So

$$\boxed{\hat{n}\hat{a}^\dagger|n\rangle = (n+1)\hat{a}^\dagger|n\rangle}$$

From Eq,(2), we may be tempted to write $\hat{a}^\dagger|n\rangle = |n+1\rangle$ so that our final result can be written as

$$\hat{n}\hat{a}^\dagger|n+1\rangle = (n+1)\hat{a}^\dagger|n+1\rangle$$

However, this definition of $|n+1\rangle$ does not fit our requirement for normalization. We can see this by assuming the $n+1$ and n state are both normalized, that is $\langle n+1|n+1\rangle = 1$ and $\langle n|n\rangle = 1$. If we have $\hat{a}^\dagger|n\rangle = |n+1\rangle$ then

$$\begin{aligned}\langle n+1|n+1\rangle &= \langle n|\hat{a}\hat{a}^\dagger|n\rangle \\ &= \langle n|\hat{a}^\dagger \hat{a} + 1|n\rangle \\ &= \langle n|\hat{a}^\dagger \hat{a}|n\rangle + \langle n|n\rangle \\ &= \langle n|n|n\rangle + 1 \\ &= n + 1 \neq 1 \qquad \text{(in general)}\end{aligned}$$

So if $\hat{a}^\dagger|n\rangle = |n+1\rangle$ then $\langle n+1|n+1\rangle = n+1 \neq 1$ and our $n+1$ state is not properly normalized. We must include some numerical factor in order to ensure the normalization of the state. So, we let $\hat{a}^\dagger|n\rangle = c_1|n+1\rangle$ where c_1 is our arbitrary numerical factor. If we go through the same calculation for $\langle n+1|n+1\rangle$ we find that

$$\langle n+1|n+1\rangle|c_1|^2 = (n+1)$$

So, in order to ensure the normalization of $|n+1\rangle$, c_1 must equal $\sqrt{n+1}$ and we have therefore have the formula

$$\boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle} \tag{3}$$

(g.) **It turns out that $|0\rangle$ is an eigenstate of $\hat{n}$ with the lowest possible eigenvalue $n = 0$ (we're going to take this as given, and skip the proof of this fact). This means that the lowest possible energy of the simple harmonic oscillator is $E_0 = \frac{1}{2}\hbar\omega$. Using (f.), construct all other eigenstates of the Hamiltonian and their corresponding eiegnenergies. In particular, show that the allowed energies and the energy eigenstates of a quantum simple harmonic oscillator are**

$$\begin{aligned} E_n &= \left(n + \frac{1}{2}\right)\hbar\omega \qquad (\textbf{where}\, n = 0, 1, 2, ..) \\ |n\rangle &= \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle \end{aligned}$$

Solution: If $|0\rangle$ is an eigenstate of the hamiltonian with eigenvalue $E_0 = \frac{1}{2}\hbar\omega$. Then by definition we have

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$$

Using Eq (3)we can construct states excited states higher than $|0\rangle$. For example,

$$\begin{aligned} |n+1\rangle &= \frac{1}{\sqrt{n+1}}\hat{a}^\dagger|n\rangle \\ |1\rangle &= \frac{1}{\sqrt{1}}\hat{a}^\dagger|0\rangle \\ |2\rangle &= \frac{1}{\sqrt{2}}\hat{a}^\dagger|1\rangle = \frac{1}{\sqrt{2\cdot 1}}\hat{a}^\dagger|0\rangle \\ |3\rangle &= \frac{1}{\sqrt{3}}\hat{a}^\dagger|2\rangle = \frac{1}{\sqrt{3\cdot 2}}\hat{a}^\dagger|1\rangle = \frac{1}{\sqrt{3\cdot 2\cdot 1}}\hat{a}^\dagger|0\rangle \\ |4\rangle &= \frac{1}{\sqrt{4}}\hat{a}^\dagger|3\rangle = \frac{1}{\sqrt{4\cdot 3}}\hat{a}^\dagger|2\rangle|2\rangle = \frac{1}{\sqrt{4\cdot 3\cdot 2}}\hat{a}^\dagger|1\rangle = \frac{1}{\sqrt{4\cdot 3\cdot 2\cdot 1}}\hat{a}^\dagger|0\rangle \end{aligned}$$

So we see that, in general

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}|0\rangle}$$

To compute the eigenvalue of the state $|n\rangle$ we apply the hamiltonian operator to our arbitrary state

$$\begin{aligned}
\hat{H}|n\rangle &= \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right)|n\rangle \\
&= \hbar\omega\left(\hat{n} + \frac{1}{2}\right)|n\rangle \\
&= \hbar\omega\left(\hat{n}|n\rangle + \frac{1}{2}|n\rangle\right) \\
&= \hbar\omega\left(n|n\rangle + \frac{1}{2}|n\rangle\right) \\
&= \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle
\end{aligned}$$

In the foruth line, we replaced the operator $\hat{n}$ with the number n because of the relation $\hat{n}|n\rangle = n|n\rangle$ stated in (f). So if we have our eigenstate-eigenenergy definition as $\hat{H}|n\rangle = E_n|n\rangle$ then we see that

$$\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$$