Large W limit of the knapsack problem

Solving the knapsack problem using statistical physics and connecting dynamic programming to greedy solutions

1 Introduction

We have N objects where object i has value v_i and weight w_i, and a "knapsack" that can hold a maximum weight of W. What is the collection of objects that has the maximum total value consistent with the weight limit?

This question is known as the "knapsack problem," and there are many standard algorithms to solve it. Two common algorithms are "dynamic programming" and "greedy solution," the former involving a recursive structure of solutions and the latter involving more myopic optimization.

2 Main Result

We represent the knapsack problem a bit more formally. We define the collection of objects in the knapsack as $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ with $x_i = 1$ if object i is in the knapsack and $x_i = 0$ otherwise, and we define our weight and value vectors as $\mathbf{w} \equiv (w_1, w_2, \ldots, w_N)$ and $\mathbf{v} \equiv (v_1, v_2, \ldots, v_N)$, respectively. Then our objective is to find $\mathbf{x}$ that maximizes $\mathbf{v} \cdot \mathbf{x}$ subject to the constraint $\mathbf{w} \cdot \mathbf{x} \leq W$. For simplicity, we take v_i and w_i to be positive integers. Thus, W is also an integer satisfying $W \geq N$. Letting $\mathbf{x}$ define the microstate of a statistical physics system at temperature $T = \beta^{-1}$, we find the partition function for this system can be written as

$$Z_N(\beta\mathbf{v}, \mathbf{w}, W) = \frac{1}{2\pi i} \oint_\Gamma \frac{dz}{z^{W+1}} \frac{1}{1-z} \prod_{k=1}^{N} \left(1 + z^{w_k} e^{\beta v_k}\right), \tag{1}$$

where Γ is a closed contour about the origin in the complex plane. From the partition function, we find the dynamic programming solution for the optimal value $V_N(W)$ for N objects and a weight limit W:

$$V_N(W) = \begin{cases} V_{N-1}(W) & \text{for } W < w_N \\ \max\{V_{N-1}(W),\, v_N + V_{N-1}(W - w_N)\} & \text{for } W \geq w_N, \end{cases} \tag{2}$$

Taking the partition function to the $W \gg w_i$ limit, we can also obtain a greedy solution for the KP:

$$V_N(W) \simeq \sum_{\ell=1}^{N} v_\ell H(v_\ell/w_\ell - \gamma_0), \quad \text{where } \gamma_0 \text{ is defined by} \quad W = \sum_{j=1}^{N} w_j H(v_\ell/w_\ell - \gamma_0), \tag{3}$$

and where $H(x) = 1$ if $x > 0$ and $H(x) = 0$ if $x < 0$. Thus, we find a relationship between these two common algorithms: