Permanents through probability distributions

Using probability distributions to compute the permanent of a matrix

1 Introduction

The permanent of a matrix is defined as

$$\mathrm{perm}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{1}$$

where the summation is over all σ that are elements of the symmetric group S_n. The permanent of a matrix, although seemingly very similar to the determinant of a matrix, is actually much more difficult to calculate. While the determinant has certain properties that make it invariant under simplifying transformations, the permanent is immune to such simplifications. Consequently, much formalism has developed over the years focused on turning the nominally $O(n!)$ calculation of Eq.(1) into a calculation of lower order.

In this work we derived a theorem which generalizes these transformations by representing the permanent as the expectation value of a probability distribution.

2 Main Result

Theorem 1 *Let $p_X : \Omega_X \to \mathbb{R}$ be a probability distribution defined over the domain Ω_X with zero mean and unit variance. Let A be an $n \times n$ matrix with elements $a_{i,j}$. Then the permanent of A is*

$$\mathrm{perm}(A) = \int_{\Omega_X^n} d^n\boldsymbol{x} \prod_{i=1}^{n} p_X(x_i)\, x_i \sum_{j=1}^{n} a_{i,j} x_j, \tag{2}$$

where $\Omega_X^n = \Omega_X \otimes \cdots \otimes \Omega_X$ is the n-factor product over the single-variable domain of integration. In condensed notation, we can write Eq.(2) as the expectation value

$$\mathrm{perm}(A) = \left\langle \prod_{i=1}^{n} x_i \sum_{j=1}^{n} a_{i,j} x_j \right\rangle_{x_i \sim p_X}, \tag{3}$$

where the average is over $\{x_i\}$, a set of independent identically distributed random variables each of which is drawn from p_X.

3 Implications

Starting from theorem 1, and choosing the probability distribution $p_X(x) = \frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)$, we are led to a preliminary form of Glynn's formula for permanents

$$\mathrm{perm}(A) = \frac{1}{2^n} \sum_{\{s_i\}} \prod_{i=1}^{n} s_i \sum_{j=1}^{n} a_{i,j} s_j, \quad \text{where} \quad \sum_{\{s_i\}} = \prod_{i=1}^{n} \sum_{s_i \in \{-1,+1\}} . \tag{4}$$

which reduces the calculation of the permanent from $O(n!)$ to $O(2^n n^2)$. If we instead take the probability distribution to be a Gaussian $\rho_X(x) = e^{-x^2}/(2\pi)^{1/2}$, we can then derive a simplified form of MacMahon's Master theorem. These two special cases suggest that Eq.(2) can serve as more general starting point for permanent calculations.