Statistical physics of the symmetric group

Computing the partition function and order parameter for a system whose microstates are elements of the symmetric group

1 Introduction

The partition functions for most simple statistical physics models (e.g., Ising Model, Ideal Gas Model, Potts Model) involve summations over a set of identically distributed independent variables. Consequently, when such systems consist of N particles, the total number of system microstates scales as V^N where V is the number of microstates available to a single particle.

But what if we considered an N particle system where the microstates available to one particle affected those available to another? A simple choice in this direction is to consider a system where microstates are the various permutations of objects in an initial list. More formally, we can say such microstates are elements of the symmetric group and analyzing the simplest such statistical physics model for such microstates yields the statistical physics of the symmetric group.

2 Main Result

Say we have a statistical physics system where the microstates are elements of the symmetric group S_N. We denote such microstates as σ and note that they map integers to integers. Formally we represent the group identification as $\sigma \in S_N$, and the mapping property as $\sigma : \mathbb{N} \to \mathbb{N}$. As an initial model, we take the energy of a particular microstate to be $\mathcal{H}_N[\sigma] = \lambda \sum_{i=1}^{N}(1 - \delta_{i,\sigma(i)})$ where $\delta_{i,j}$ is the Kronecker delta and $\lambda > 0$. In other words, across all i for every $\sigma(i)$ that does not map to i, there is an energy cost of $+\lambda$ in the system. This energy cost should lead to the system settling into the identity mapping microstate ($\sigma(i) = i$ for all i) at low temperatrues. Computing the partition function for this system gives us

$$Z_N(\beta\Delta) = \sum_{\sigma \in S_N} \exp\left(-\beta \mathcal{H}_N[\sigma]\right) = \int_0^\infty dx\, e^{-x} \Big(1 + (x-1)e^{-\beta\Delta}\Big)^N, \tag{1}$$

where the discrete summation is over all elements of the symmetric group. The order parameter for this system is $\langle j \rangle = \langle \sum_{i=1}^{N}(1-\delta_{i,\sigma(i)})\rangle$, which represents the average number of deranged elements in the system where a "deranged element" is one for which $\sigma(i) \neq i$. Considering the integral in Eq.(1) in the $N \gg 1$ limit, we find a temperature dependent expression for $\langle j \rangle$:

$$\langle j \rangle = N - e^{\beta\Delta}. \tag{2}$$

Taking $T \to \infty$ in Eq.(2) yields $\langle j \rangle \simeq N$ meaning that at large temperature (where entropy dominates) the system occupies the most deranged microstates. Conversely, Eq.(2) suggests that below the temperature $k_B T_c = \Delta / \ln N$, the system settles into the most ordered microstate in which $\sigma(i) = i$ for all i.

3 Implications

With the basic state space presented above we can consider increasingly complex Hamiltonians and systems. Defining $j = \sum_{i=1}^{N}(1 - \delta_{i,\sigma(i)})$, we can consider the new energy function $\mathcal{E}(j) = \lambda_1 j + \lambda_2 j^2/2N$. This energy function yields a more complex order parameter than that in Eq.(2), and we find that the associated system has two critical temperatures and multiple phase coexistence boundaries. The fact that such a simple energy function yields such complex thermal behavior suggests that having a "non factorizable" state space (like one defined by permutations of a list) results in thermal behavior that can't be encompassed by the simple statistical models whose total microstate space is just a product of the spaces of its constituent particles.