Duality and Description

These notes[1] are part of a series concerning "Motifs in Physics" in which we highlight recurrent concepts, techniques, and ways of understanding in physics. In these notes we discuss the concept and role of duality: Having two mutually valid, but distinct, physical descriptions of a system.

The concept of duality

For systems which are said to exhibit duality, the basic idea is that there is a way to mathematically model the system in two ways each of which is independent of the other and both of which are valid. Moreover, there must be a way to transform one model into the other and applying the transformation twice should return us to the original model. Thus, in a duality, we have two ways to describe a physical system in addition to a dictionary (which is an **involution**[2]) which maps between them.

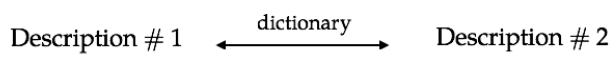

Figure 1: Schematic representation of a duality. In order to have a duality one needs two mutually valid and independent physical models or descriptions in addition to a dictionary translating between them. Applying the translation twice returns us to the original description.

Modeling physical systems can often be difficult because the physical variables to which we have access are not always the variables we need for our models. So, having two physical descriptions of the same system, where each description is defined by a different set of dynamical variables, affords us additional flexibility in analyzing the system.

However, a general accounting of dualities in physics can become quite confused because the word "duality" has been known to refer to rather different things. Historically, duality has referred to:

1. **Duality between variables:** When it is possible to exchange sets of dynamical variables in a system while leaving the dynamical equations unchanged.

2. **Duality between physical formalisms:** When it is possible to reformulate the calculational apparatus of a theory by computing the Legendre transform of a physical quantity.

3. **Duality between systems:** When two systems can be shown to be equivalent when each one is considered in the appropriate limiting case.

Still, in spite of these differences, all dualities have the common properties mentioned above: they consist of two ways of characterizing a physical system along with a dictionary to map between these two ways, and, if the mapping is applied twice, we return to the original physical characterization. In the subsequent sections, we will discuss the various types of dualities as they arise in physics, give toy-examples that exhibit basic features of the dualities, and then discuss real examples of dualities in physics. Our examples are drawn from classical physics and modern physics and we end by describing the "Wave-particle duality" which is the most famous duality, but does not fit our formal definition of duality.

[1] Inspired by a blog post by Philip Tanedo

[2] A function or transformation f which takes x to $f(x)$ is an involution if it is its own inverse, namely, if

$$f\big(f(x)\big) = x. \tag{1}$$

Duality between variables

A duality exists between variables when it is possible to transform sets of variables into one another while leaving the dynamical equations unchanged.

Such transformations are considered "dualities" rather than symmetries because, firstly, these transformations do not form a group under which infinitesimal variants exist, and secondly, these transformation move between distinct dynamical variables rather than different forms of a single dynamical variable.

Often, the variables that we use to define a physical system are the ones which, historically, have been the easiest to measure. However, dualities between variables help us realize there are more fundamental ways to characterize a system beyond our biases established by experimental accessibility.

Toy Example: Variables of an Oscillator

It is easy to construct a simple model of this type of duality. Say we have the following dynamical equations relating the time evolutions of R and L:

$$\dot{R}(t) = -\frac{\Omega}{\lambda} L(t) \tag{2}$$

$$\dot{L}(t) = \Omega\lambda\, R(t). \tag{3}$$

If we make the transformations

$$L(t) \to -\lambda R(t) \quad \text{and} \quad R(t) \to L(t)/\lambda, \tag{4}$$

then the two equations are left unchanged. Thus, we can say there is a duality between $L(t)$ and $R(t)$. Indeed, taking both equations together leaves us with the identical oscillator equations

$$\ddot{R}(t) = -\Omega^2 R(t) \qquad \text{and} \qquad \ddot{L}(t) = -\Omega^2 L(t). \tag{5}$$

We can then use either $R(t)$ or $L(t)$ when characterizing the dynamics of the system governed by Eq.(2) and Eq.(3).

Example: Electro-Magnetic duality

Historically, the first duality of field theory physicists discovered was the one relating electric and magnetic fields. This duality is seen most transparently in the source-free Maxwell equations which exhibit symmetrical dynamics for the electric and magnetic fields. By applying the correct transformation, we can convert the equations governing the spatial and temporal evolution of the electric field into those governing the spatial evolution of the magnetic field and vice versa (See Fig. 2). This duality is not at all incidental and in fact what we distinguish as the electric and magnetic fields stems from the bias of our particular reference frame.

Electric Field

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\mathbf{B} \to -\frac{1}{c}\mathbf{E}$ ⟷ $\mathbf{E} \to c\mathbf{B}$

Magnetic Field

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Figure 2: Duality of Maxwell's Equations

In much the same way that special relativity tells us that the distances and time intervals we measure depend on the frame in which we measure them, what we define as the electric and magnetic fields depends on how we're moving relative to the charge and current distributions which create them (See chapter 5 of [1] for a discussion).

Duality between formalisms

A duality exists between physical formalisms when it is possible to take the Legendre transform of the physical quantity (often an energy or some generalized potential) foundational to one formalism and obtain the physical quantity foundational to the other formalism.

This type of duality was discovered in mechanics in the mid-19th century, and it is thus, historically, the first duality of physics. However, it is not often seen as a duality today since the term "duality" is usually reserved for the type of duality that we will discuss in the next section. However, since Legendre transforms provide a precise dictionary between functions and because the Legendre transform of a Legendre transform is a net identity operation (and hence the Legendre transform is an involution transformation), we can consider physical formalisms related through Legendre transforms as duals of one another.

Toy Example: Energy and Co-energy of Capacitor Plates

As a simple example of duality between formalisms, we can consider energy and "co-energy" [2] in electrodynamics.

Say we have two capacitor plates separated by a distance x and with total capacitance $C(x)$. We want to compute the force between these capacitor plates, but we want to do so from two perspectives. By one perspective, the charge Q on the capacitor plates is taken to be held at a constant value, and the potential V between the plates changes as we change x. In the second perspective, the potential V between the two plates is held constant, and the charge Q changes as we change x. Regardless of what perspective we choose, we should find the same force and hence the same dynamics for each potential. In this way, the two ways to conceptualize how charge and potential energy flow between the plates are dual to one another.

First, we define the energy of these capacitor plates from the first perspective. For plates of capacitance $C(x)$ and a charge Q on each plate, the potential energy is

$$U(Q,x) = \frac{Q^2}{2C(x)}. \tag{6}$$

Adding charge to the system increases the energy due to the potential difference between the two plates, and increasing the distance between the two plates increases the energy due to the attractive force between the two plates. Thus, the differential of the potential energy U is

$$dU = V dQ - F dx, \tag{7}$$

from which we can infer

$$\frac{\partial}{\partial Q} U(Q,x) = V \qquad \text{and} \qquad \frac{\partial}{\partial x} U(Q,x) = -F. \tag{8}$$

Eq.(6) is the potential energy under the assumption that Q is the independent variable and that V (according to the capacitance-charge-potential equation) with changes with x.

Computing the force from Eq.(8), we find

$$F = -\frac{\partial}{\partial x} U(Q, x) = \frac{Q^2}{2C(x)^2} \frac{\partial}{\partial x} C(x), \tag{9}$$

which tells us how much force each capacitor plate exerts on its connecting wire.

Now, we will compute this force using the ”co-energy” of the capacitor plates, i.e., the Legendre transform of Eq.(6) expressed in terms of V and x. Computing the differential of VQ, we can show

$$V dQ = d(VQ) - Q\, dV. \tag{10}$$

Inserting this expression into Eq.(7), we find

$$dU^* = Q\, dV + F dx, \tag{11}$$

where we defined

$$U^*(V, x) = VQ(V) - U\big(Q(V), x\big). \tag{12}$$

$Q(V)$ is the charge on a capacitor plate as a function of the potential difference between the two plates. It is obtained by inverting the equation in the first equality of Eq.(8). Eq.(12) defines the Legendre transform of $U(Q, x)$ and is called the ”co-energy” (or co-potential energy) of the capacitor. This co-energy has a relationship with force different from the standard one. From Eq.(11), we can infer

$$\frac{\partial}{\partial V} U^*(V, x) = Q \qquad \text{and} \qquad \frac{\partial}{\partial x} U^*(V, x) = F, \tag{13}$$

indicating that, for this co-energy, the force is found simply by taking the derivative. Computing Eq.(12) explicitly, gives us

$$U^*(V, x) = \frac{1}{2} V^2 C(x). \tag{14}$$

For $U^*(V, x)$, the potential V is taken to be the independent variable with the charge Q changing (according to the capacitance-charge-potential equation) with changing x. Computing the force associated with this co-energy using Eq.(13), we obtain

$$F = \frac{\partial}{\partial x} U^*(V, x) = \frac{1}{2} V^2 \frac{\partial}{\partial x} C(x). \tag{15}$$

Comparing Eq.(9) and Eq.(15), and using $Q/C = V$, we see that the force computed from the two formalisms is the same. Thus, although U^* and U amount to dual perspectives on whether charge or potential energy changes with changes in x, the two perspectives lead to the same dynamics and hence the same physical results.

Example: Hamiltonian-Lagrangian duality

Students typically begin their study of classical mechanics with Newton’s laws. These laws are intuitive and devoid of any difficult to visualize mathematical abstractions, but as one attempts to move from classical mechanics to other physical theories like statistical mechanics or quantum mechanics, Newtonian mechanics proves to be a poor starting point. This is because it uses concepts which do not have clear analogs when we are not trying to model particle motions. Fortunately, there are two related formulations of mechanics which are easier to extend because they are framed around the more generalizable concepts of energy and action.

In analytical mechanics, we have a choice of whether to model a physical system with positions and velocities or with positions and momenta. Using positions and velocities places us within the **Lagrangian formalism** of mechanics where the equations of motion are defined by a system of second-order differential equations, while using positions and momenta[3] places us within the **Hamiltonian formalism** of mechanics where the equations of motion are a system of coupled first-order differential equations.

Hamiltonian Mechanics	$\mathcal{L} = p\frac{\partial \mathcal{H}}{\partial p} - \mathcal{H}$	Lagrangian Mechanics
$q_i, p_i,\ \mathcal{H}(\{q_i, p_i\})$	$\longleftrightarrow$	$q_i, \dot{q}_i,\ \mathcal{L}(\{q_i, \dot{q}_i\})$
	$\mathcal{H} = \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L}$	

Figure 3: Duality of Classical Mechanics

Because the two formalisms yield the same dynamics, they are equivalent descriptions of a physical system. The dictionary which allows us to transform from the Lagrangian formalism to the Hamiltonian formalism is the Legendre transform (See Fig. 3). Each of these formalisms has their own advantages outside of classical mechanics. Hamiltoniains are useful in moving from classical mechanics to quantum mechanics and statistical mechanics, whereas Lagrangians are useful in moving from classical field theory to quantum field theory.

Example: Counting of States-Partition Function duality

When studying the thermal equilibrium properties of a system with a fixed number of particles, we can choose how to characterize the system contingent on which parameters are constant. For a system with a constant energy E, all the microstates of the system are equally probable and we study the system with what is known as the **microcanonical ensemble**. Alternatively, if the system is at a constant temperature T, then we study the system with what is known as the **canonical ensemble**. In the microcanonical ensemble the defining thermodynamic potential is the entropy $\mathcal{S}$ written in terms of the number of microstates $\Omega_N(E)$ of the system as $\mathcal{S} = \ln \Omega_N(E)$. In the canonical ensemble the defining thermodynamic potential is the free energy $\mathcal{F}$ written in terms of the partition function $Z_N(\beta)$, where $\beta = 1/k_B T$ as $\mathcal{F} = -\ln Z_N(\beta)$[4]. In statistical mechanics, we know that $Z_N(\beta) = \int_0^\infty dE\, \Omega_N(E) e^{-\beta E}$. This definition allows us to show that the free energy of the canonical ensemble and the maximized entropy of the microcanonical ensemble are related through a Legendre transform.

Microcanonical Ensemble	$\mathcal{F} \simeq E\frac{\partial}{\partial E}\mathcal{S} - \mathcal{S}$	Canonical Ensemble
$\Omega_N(E)$, Constant E	$\longleftrightarrow$	$Z_N(\beta)$, Constant β
$\mathcal{S} = \ln \Omega_N(E)$	$\mathcal{S} \simeq \beta\frac{\partial}{\partial \beta}\mathcal{F} - \mathcal{F}$	$\mathcal{F} = -\ln Z_N(\beta)$

Figure 4: Duality of Statistical Mechanics. We use $\simeq$ instead of full equality because this duality arises from applying steepest descent to the Laplace transform which expresses Z_N in terms of Ω_N.

[3]In mathematical parlance, we say the positions and velocities are in the tangent bundle of particle trajectories and positions and momenta are in the cotangent bundle. The tangent and cotangent bundles contain vector spaces dual to one another and thus we can see the set of positions and velocities as dual to the set of positions and momenta. Therefore, the mathematical motivation for the above expressed duality is that two functions (the Hamiltonian and the Lagrangian) of mutually dual coordinates are dual physical models of one another.

[4]We use dimensionless definitions of entropy and free energy because they simplify the Legendre transform in Fig. 4.

The idea that the canonical ensemble and the microcanonical ensemble are on two sides of dual formalisms of statistical mechanics is not a typical one, but given that the free energy and the maximized microcanonical-ensemble entropy are related through a a mathematical transformation which relates dual descriptions of a system, i.e., the Legendre transform, this interpretation is consistent with our definition of duality in Fig. 1 [3].

Duality between systems

A duality exists between two systems when the physical properties of the one system, considered in a certain energy or parameter regime, reduce to the physical properties of the other system in a different energy or parameter regime. In a modern context, this type of duality is often what physicists (primarily high-energy and condensed-matter theorists) are referring to when they claim a theory exhibits a ”duality”.

Toy Example: Strong and Weak Coupling of Partition Functions

Consider the following toy model. Say we have two partition functions $Z_1(\lambda)$ and $Z_2(g)$ which have the analytic forms

$$Z_1(\lambda) = e^{\lambda} \int_1^{\infty} dt \, \frac{e^{-t\lambda}}{t} \, , \qquad Z_2(g; N) = \sum_{n=0}^{N} (-1)^n n! \, g^{n+1} \tag{16}$$

where $\lambda, g > 0$. We take the two partition functions in Eq.(16) to describe two different physical systems. Depending on whether you want to take these functions to be partition functions as they occur in statistical mechanics or toy-analogs of the functional integrals in QFT, you could, respectively, consider λ and g to both be inverse functions of temperature or to both be coupling constants.

We will now explore a large λ-small g duality between $Z_1(\lambda)$ and $Z_2(g)$. Applying integration by parts successively to $Z_1(\lambda)$ we obtain

$$Z_1(\lambda) = \frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{2!}{\lambda^3} - \frac{3!}{\lambda^4} + \cdots \, . \tag{17}$$

This series is divergent, but we can truncate it and use it as an approximation for $Z_1(\lambda)$ for sufficiently large λ. Conversely, for sufficiently small g, the partition function $Z_2(g; N)$ becomes

$$Z_2(g; N) = g - g^2 + 2! g^3 - 3! g^4 + \cdots \, . \tag{18}$$

Comparing Eq.(17) and Eq.(18), we see that the two partition functions are equivalent for large λ and small g. More specifically, we have

$$Z_1(\lambda \gg 1) \simeq Z_2(g \ll 1; N). \tag{19}$$

The dictionary is as follows: If we start from $Z_1(\lambda)$ and we want to know the large λ behavior of the associated system, then we evaluate $Z_2(g; N)$ with $g \ll 1$, and we then we make the transformation $g \to 1/\lambda$. The reverse transformation can be similarly described, and applying the transformations twice give us $\lambda \to \lambda$ or $g \to g$ as we see in involutions.

Modern Dualities

We consider the duality defined in Eq.(19) as a toy-example of a strong-weak duality because the ”strong coupling” (i.e., $\lambda \gg 1$) limit of one theory is shown to be equivalent to the ”weak coupling” (i.e., $g \ll 1$) limit of another theory. The utility of this type of duality exists in the way this relationship allows perturbation theory to be applied to nonperturbative systems: Since systems with strong coupling cannot be analyzed using general perturbation theory methods, being able to transform such systems into weakly coupled versions greatly increases our ability to study them.

In the 20th century, physicists have discovered many dualities of this type and all of them are geared towards making analytically difficult problems more tractable.

- **Gauge-Gravity duality:** This is a specific form of a general class of conjectures framed around the "holographic principle". The holographic principle posits that for some physical systems in a definite volume, the physics of the system can be characterized by properties at the boundary of that volume. Similarly, the gauge-gravity duality posits that certain theories of gravity in d dimensions are equivalent to certain quantum field theories in $d-1$ dimensions. The duality has allowed physicists to study abstract models of strongly coupled dualities by studying their analogous representation in the weakly coupled gravity theory. It is more formally known as the **AdS/CFT correspondence**.

- **Strong-Weak coupling duality:** This is a generalization of the electro-magnetic duality discussed in the first section. It states that the properties of the weakly interacting electric charges and strongly interacting magnetic monopoles of one gauge theory (which can be seen as a generalization of electrodynamics) are equivalent to the strongly interacting electric charges and weakly interacting magnetic monopoles of the dual theory. Magnetic monopoles have never been observed, so this duality is mostly used to explore the theoretical properties of toy model gauge theories. This duality is more formally known as **S-duality**.

- **High T-Low T duality:** For spins arranged in a two-dimensional square lattice, one finds that the high temperature expansion of the partition function can be transformed into the low temperature expansion of the partition function. Employing both expansions together allows one to compute the critical temperature for the two-dimensional square Ising model much more simply than in the standard derivation. This duality is more formally known as the **Kramers-Wannier duality**.

Aside: Wave-Particle Duality

Anyone who has studied quantum mechanics has heard of the wave-particle duality. As far as dualities go, the wave-particle duality is mostly a qualitative one stating that, on the subatomic level, "particles" like electrons and photons have both wave-like and particle-like properties and cannot be exclusively interpreted as either waves or particles. Thus the wave-particle duality is not a true duality because rather than positing the equivalence between two mathematical formalisms, it concerns how best to interpret the single mathematical formalism of position-space quantum mechanics.

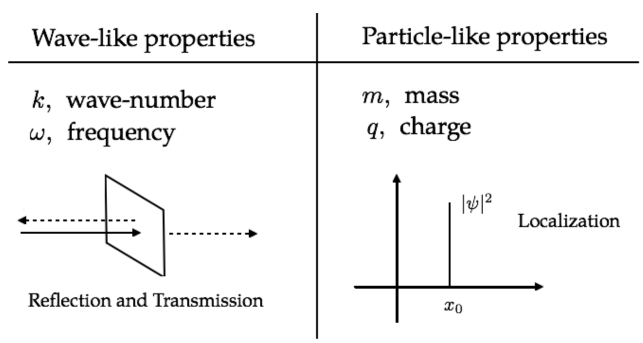

Figure 5: Wave-particle "duality" states that quantum systems have both wave-like and particle-like properties.

The wave-particle duality was first suggested by Louis de Broglie in his dissertation and was made more concrete by Schrödinger through the development of his wave equation. From the Schrödinger equation, it is apparent that although we can describe subatomic particles by specific masses and charges, these particles can also be characterized by wave numbers and frequencies and can be delocalized like waves.

Given the fact that this duality does not fit into the standard picture of dualities given in Fig. 1—namely, there are not two equivalent descriptions of a physics system nor is there a dictionary mapping between them—physicists today think it is better to rename the duality ”wave-particle complementarity” which contends that quantum systems cannot be well characterized by choosing a single side in the wave-particle dichotomy; one always needs both perspectives to characterize the system.

What’s the point?

What is the point of having two ways to model a physical system or having one physical model which is equivalent to another of a completely different physical regime? Dualities in general provide us with greater flexibility in studying and applying physical theories and thus supplement our understanding of the systems they model. Moreover, having two ways to model a physical system, both ways of which yield consistent answers suggests there is a deeper physical mechanism or principle at work for which the two ways are primarily special cases.

References

[1] E. M. Purcell, *Electricity and magnetism*, vol. 2. McGraw-Hill New York, 1965.

[2] J. R. M. Herbert H. Woodson, *Electromechanical Dynamics, Discrete Systems*, vol. 1. Krieger Publishing Company, 1990.

[3] R. K. Zia, E. F. Redish, and S. R. McKay, “Making sense of the legendre transform,” *American Journal of Physics*, vol. 77, no. 7, pp. 614–622, 2009.