Importance of Geometry

These notes are part of a series concerning "Motifs in Physics" in which we highlight recurrent concepts, techniques, and ways of understanding in physics. In these notes we discuss how physics has been (and continues to be) influenced by geometry.

The First language of Physics

Although Newton is often credited with developing both calculus and the foundations of modern mechanics, he kept the two largely separate in the *Principia*. That is, instead of using calculus to prove many of the theorems concerning orbits and planetary motion (as is done in a mechanics course today) Newton worked geometrically and developed his arguments in the tradition of proofs found in Euclid's *Elements*.

For example, in proving that central forces lead to orbits which sweep out equal areas in equal times Newton employed a geometric proof of the kind in Fig. 1[1], rather than the calculus-based proof grounded in angular momentum conservation.

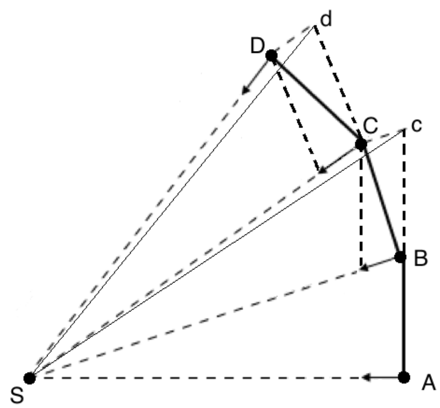

Figure 1: Geometric Proof of Kepler's First Law: The star (the origin of the central force) is at S and the planet moves sequentially from A to B to C to D in equal times.

Newton's use of geometry in the *Principia* is perhaps not terribly surprising since geometry was the first axiomatic and rigorously developed field of mathematics. Newton likely knew that a mathematically inclined natural philosopher reading his work would have lent greater credibility to a formalism grounded in geometry rather than in some less well understood new mathematical theory[2].

But it is nevertheless noteworthy that many of the foundational elements of mechanics, now understood through calculus and algebra, were initially expressed geometrically. On one level it is a reflection of the fact that geometry has both a visual and analytical representation. On another level it reveals that since the inception of modern theoretical physics, geometry has been crucial to the building and systematizing of the discipline. Indeed, if physical laws are written in the language of mathematics, then geometry was the first language of physics, and even long after Newton's time, geometry continues to have a great, if often unarticulated, influence on physics.

[1] Given that $Cc \parallel SB$, $Dd \parallel Sc$, $AB = Bc$, and $BC = Cd$, we can show that the triangles SAB, SBC, and SDC have equal areas.

[2] This is my historically unrigorous speculation. There is apparently some debate as to why Newton didn't use calculus to derive his results in the *Principia* [1].

Geometry of Phase Space

In our notes on the evolution of time, we characterized physical systems by their dynamics, kinematics, and the configuration spaces through which variables in the systems evolved. In classical systems, configuration space is the space of possible values of the dynamical variables. However, because classical systems are governed by second-order equations, they are defined by two degrees of freedom per particle per dimension. Thus in addition to configuration space, classical systems are defined by a momentum space, the space of possible values of the variables canonically conjugate to the dynamical variable. Together, the configuration space and momentum space define what is known as the **phase space** of our system.

In classical systems there is an important result which underlies statistical mechanics and concerns the geometry of our phase space. Say we have a large number of particles which comprise a volume in our phase space, and we ask how does the phase-space volume these particles occupy change in time? This question might seem arbitrary, but it is actually a crucial one because if we find (as we do) that certain properties of our system's phase space are independent of time, then we could ignore the specific time dynamics of those properties.

This is precisely what **Liouville's theorem** asserts: for energy conserving systems, the volume of phase space the system occupies does not change in time. We depict this theorem schematically in Fig. 2.

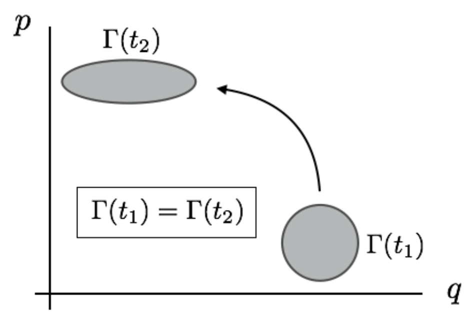

Figure 2: Visual depiction of Liouville's Theorem: As our system evolves in time, the phase-space "volume" (area in this case) our system occupies does not change (i.e., $\Gamma(t_1) = \Gamma(t_2)$).

Statistical physics provides coarse-grained models of systems with a very large number of degrees of freedom. In statistical physics, we find that the probability to be in a particular microstate of the system is defined by the volume of phase space corresponding to the energy of that microstate. Therefore, Liouville's theorem asserts that as a system with a large number of degrees of freedom evolves, the relative volumes of various regions of the phase space (and hence the relative probabilities to be in those regions) does not change either. From Liouville's theorem, physicists have postulated the **ergodic hypothesis**, a fundamental assumption of statistical physics which posits that time averages of thermal systems can be equated to phase-space (or ensemble) averages. In mathematical form, the ergodic hypothesis can be written as shown in Fig. 3.

$$\langle O \rangle = \lim_{T\to\infty} \int_0^T dt\, O\left(\mathbb{P}(t), \mathbb{Q}(t)\right) = \int d\mathbb{P}\, d\mathbb{Q}\, O(\mathbb{P}, \mathbb{Q})\, \rho(\mathbb{P}, \mathbb{Q}) \qquad \text{[Ergodic Hypothesis]} \tag{1}$$

Figure 3: Ergodic Hypothesis: O is an observable we want to average over time. The time average $\langle O \rangle$ is equivalent to the ensemble average over $\mathbb{P}$ (i.e., momentum) space and $\mathbb{Q}$ (i.e., configuration) space weighted by the density $\rho(\mathbb{P}, \mathbb{Q})$ of the phase space.

From the ergodic hypothesis, we can derive the canonical ensemble and all the other important ensembles of statistical physics. Thus, retracing the argument, we see that our understanding of how the geometrical properties of phase space change (or rather, do not change) in time allows us to study equilibrium thermodynamics, a naively time-dependent theory, with no direct reference to time at all.

Geometrical Notations of Electrodynamics

Despite what many people (mathematicians included) think, much of the evolution of mathematics consists not merely of developing new mathematical formalisms and concepts but also in finding new ways to clearly and cogently express existing formalisms. These new ways of expressing old ideas then contribute to the spread and further development of mathematics by allowing these ideas to be effectively employed by those who did not discover them in the first place. Thus largely due to improvements in pedagogy, calculus, which was cutting-edge research mathematics in the 17th and 18th century, can nowadays be learned and applied by adolescents who spend most of their time thinking about other things.

Similarly, in the mid-to-late 19th century, the mathematical theory of electrodynamics was developed primarily by and for physicists and mathematicians[3], and it only spread to engineering after Heaviside and Gibbs [2] developed the vector analysis language which gave it its modern form. When James Clerk Maxwell first composed his *Treatise on Electricity and Magnetism* [3], he did not employ modern vector analysis notation[4] to express mathematical results in the theory. For example, when writing the formula which defines the properties of the magnetic field, Maxwell would write, in full component glory,

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0\,, \quad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x\,, \quad \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_y\,, \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_z, \tag{2}$$

rather than the more compact forms

$$\nabla \cdot \vec{B} = 0\,, \quad \nabla \times \vec{B} = \mu_0 \vec{J}, \tag{3}$$

which only came about after Heaviside and Gibbs's work.

One could reasonably claim that the notational change in going from Eq.(2) to Eq.(3) is a trivial one which contributes marginal additional insight to the phenomena of magnetism. However, as is generally true, this notational change introduced the possibility of extending electromagnetism beyond the context of its formulation. Given that it appears we live in a world with three spatial dimensions, it makes sense that the laws of electromagnetism bear explicit reference to this dimensionality. But considering the possibility of extra dimensions would require that we generalize Maxwell's equations in a way which is not at all obvious in the cartesian representation of Eq.(2). Conversely, using differential geometry, the vector analysis representation Eq.(3) can be more easily extrapolated to what is known as an **exterior derivative**, a generalized derivative which consolidates the notion of gradient, divergence, and curl. The end result is that all of Maxwell's equations (not just the two in Eq.(3)) can be written in the compact and fully general form (which makes no reference to dimensionality),

$$\mathbf{dF} = 0\,, \quad \mathbf{d} \star \mathbf{F} = \mathbf{J}, \tag{4}$$

where **d** is the exterior derivative, **F** is the field two-form, **J** is the current three-form, and $\star$ is the hodge dual operator. [5]

[3]Of course, experimentalists at the time, like Michael Faraday, did much work in exploring the experimental aspects of the theory.

[4]However, a very similar formalism of "quarternions" did exist and was applied by Maxwell in the treatise.

[5]See Arapura's Introduction to Differential Forms" ➡ [4] for an elementary discussion of differential forms. See Sjamaar's "Manifolds and Differential Forms" ➡ [5] for a less elementary discussion of differential forms.

Duality between Geometry and Energy

For much of its early history, mathematical physics was concerned with understanding how physical quantities evolved in a fixed background of space and time. But with Einstein's theory of relativity, space and time were consolidated into what was ever after known as spacetime and which, far from being the pristine and static stage for the principal actors of the physical world, became an actor itself and was postulated to evolve and change according to its own physical laws.

Our above discussions on the relevance of geometry to physics mostly involved how the mathematical representations of our physical laws contained geometric undertones. But with general relativity, these undertones are made explicit by promoting geometry to a defining feature of our spacetime. In relativity, large energy densities lead to large curvatures of spacetime, so that what we experience as the gravity created by a massive object is "really" just our motion within the curved spacetime caused by the object's energy density.

$$\text{Energy density} \quad \Longrightarrow \quad \text{Curved spacetime} \qquad \text{[General Relativity]} \tag{5}$$

But again in the early 20th century, there developed another physical discipline which posited a relationship between the geometry of spacetime and the energy contained within it. Quantum physics is not typically seen as a subject which would cause us to reinterpret our understanding of space, but such a reinterpretation is certainly present in the formalism underlying the Schrödinger equation. When we study quantum physics in position space, our goal is to find the energy eigenfunctions and energy eigenvalues given a certain potential energy. So in a way which is obvious, the energy eigenvalues we find depend on our system's potential energy. However, in a way which is less obvious, but is nonetheless true (and arguably more profound), our energy eigenvalues depend on the properties of the space which defines our system.

$$\text{Space/spacetime} \quad \Longrightarrow \quad \text{Energy spectrum} \qquad \text{[Quantum Physics]} \tag{6}$$

As a simple example of this fact consider a free quantum particle of mass m. If the particle is allowed to move throughout all of the x domain (from $x = -\infty$ to $x = \infty$), then solving the Schrödinger equation would yield energy eigenvalues

$$E_k = \frac{\hbar^2 k^2}{2m}, \tag{7}$$

where k takes on continuous values. However, if our particle could only exist within a finite domain (from $x = 0$ to $x = a$), then we would find the same energy eigenvalues Eq.(7) except k would take on discrete values.

This general relationship between space and energy appears as well in relativistic quantum physics. There are typically nuances associated with extrapolating Schrödinger mechanics to a relativistic context[6], but ignoring such subtleties, we find that the relativistic Schrödinger equation (also known as the **Klein-Gordon Equation**) in flat spacetime yields a continuous spectrum of energy values, but solving the relativistic Schrödinger equation equation in a spacetime with constant negative curvature (known formally as **AdS spacetime**) yields a discrete spectrum of energy values.

In both of these quantum mechanical examples, we change the properties of space (or spacetime) in our system and discover that the energy spectrum changes. Conversely, in general relativity when we change the energy density of our system, the spacetime changes (See Fig. 4).

[6] See early chapters of [6] for a historical discussion.

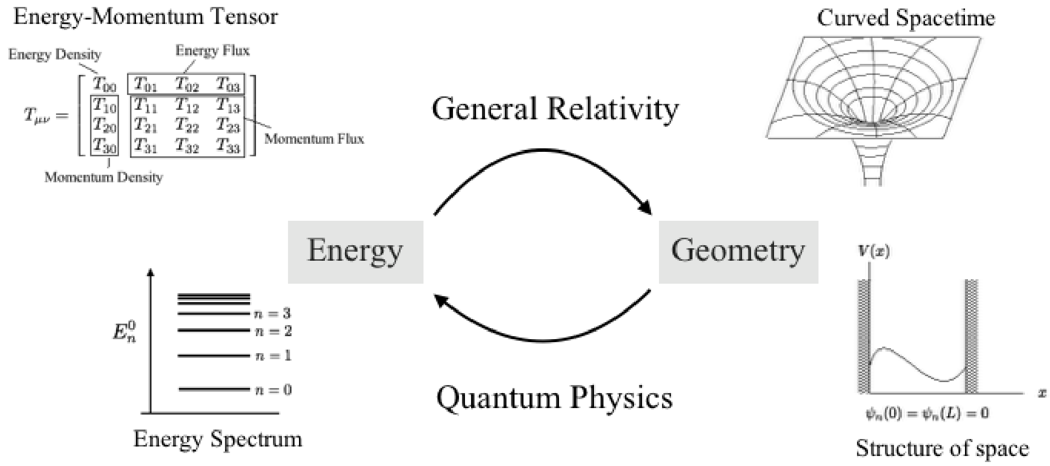

Figure 4: Duality between geometry and energy

In essence the physics of systems on large scales and the physics of systems on very small scales seem to have correspondingly dualistic relationships with energy and geometry. Thus, it perhaps is no surprise that attempt to find theories of quantum gravity require new interpretations of both geometry and energy in physics.

References

[1] N. Guicciardini, "Did newton use his calculus in the principia?," *Centaurus*, vol. 40, no. 3-4, pp. 303–344, 1998.

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[3] J. C. Maxwell, *A treatise on electricity and magnetism*, vol. 1. Clarendon press, 1881.

[4] D. Arapura, "Introduction to differential forms,"

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[6] S. S. Schweber, *QED and the men who made it: Dyson, Feynman, Schwinger, and Tomonaga*. Princeton University Press, 1994.