Oscillations, Superpositions, and Perturbations

These notes are part of a series concerning "Motifs in Physics" in which we highlight recurrent concepts, techniques, and ways of understanding in physics. In these notes we discuss how the linearity of dynamical equations is related to oscillations, superposition principle, and perturbation theory.

Equations of Physics

A student once told me that physics was "just mathematics" and that he expected to do well in the former because he always did well in the latter. As the student walked blithely away confident in his future performance, I struggled to articulate why his first assertion didn't seem quite right. It seemed to be a conflation between what is necessary and what is sufficient, a confusion between what is a property of a thing and what is an adequate definition.

Now, it is true that physics uses mathematics and one needs to understand some mathematical ideas in order to understand physics. However, physics cannot be completely reduced to mathematics because the premises and principles of physical disciplines are inevitably grounded in an interpretation of physical reality, rather than an interpretation non-material logic.

And yet these physical principles are not arbitrary and throughout all areas of physics their mathematical representations have a common form. Namely, if $F(t, \mathbf{x})$ is the dynamical function of interest in our system (e.g., a vector in Hilbert space or a classical field), then the equation which governs $F(t, \mathbf{x})$'s dynamics in space and/or time often has the form

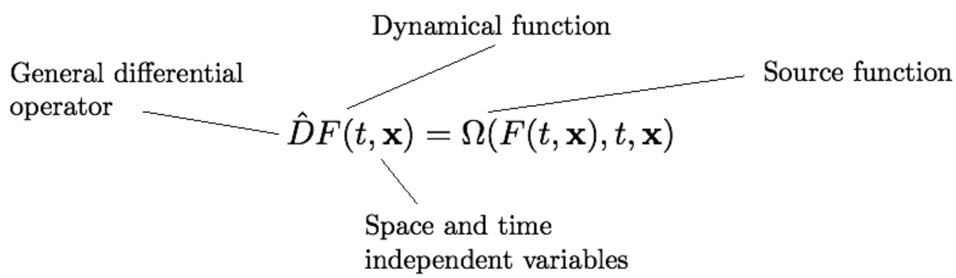

Figure 1: General form of dynamical equation is physics. The differential operator $\hat{D}$ is, most generally, a linear combination of derivatives.

The source function $\Omega(F(t, \mathbf{x}); t, \mathbf{x})$ is responsible for generating the non-trivial dynamics of $F(t, \mathbf{x})$. It can be a function of $F(t, \mathbf{x})$ or a function of some other variable which has a dynamical equation of its own. We should note that $\hat{D}$ cannot in practice be *any* linear combination derivatives, but rather its form is constrained by fundamental symmetries of physics and by the independent variables relevant in our system. Also the source function $\Omega(F, t, \mathbf{x})$, in certain physical regimes, can have derivative operator terms and nonlinear terms which are physically dominant over the term $\hat{D}F$.

As examples, some equations and their associated field of physics which bear the form in Fig. 1 are

– Gauss's Law (Electrodynamics)

$$\nabla \cdot \mathbf{E}(t, \mathbf{x}) = \rho(t, \mathbf{x}). \tag{1}$$

– Klein Gordon Equation (Classical Field Theory)

$$\left(c^{-2}\partial_t^2 - \nabla^2 - m^2c^2/\hbar^2\right)\phi(t, \mathbf{x}) = J(t, \mathbf{x}). \tag{2}$$

– Schrödinger Equation (Quantum Mechanics)

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{3}$$

Therefore we see that the form of dynamical equations shown in Fig. 1 can itself be seen as a physical motif because it appears often throughout many fields of physics. But, more deeply, this form is connected to three other common features of physical systems.

Oscillations, Superpositions and Perturbations

Related to this general form of dynamical equations are three mini-motifs which occur throughout physics:

◦ **The Harmonic oscillator is everywhere**

The universal form of the fundamental equations of physics Fig. 1 leads to the ubiquity of a particular physical model: the harmonic oscillator. In virtually every field of physics, a harmonic oscillator system is studied on some level, and it is considered so archetypal and foundational to physics that some physicists consider theoretical physics to be the practice of analyzing the harmonic oscillator at ever-higher levels of abstraction[1].

More generally, not only are oscillating system ubiquitous throughout physics but so too are decaying systems, and together these two types of phenomena result from a common property of fundamental equations of the form in Fig. 1. For such equations, if we study the system they model at low energies or near their equilibrium (i.e., constant in time) configurations, we can linearize the equation such that the dynamical variable $F(t,\mathbf{x})$ appears at most to first order[2]. Namely, we obtain

$$\left[\hat{D} - \Omega_{(1)}(F_0,t,\mathbf{x})\right]F(t,\mathbf{x}) = \Omega_{(0)}(F_0,t,\mathbf{x}) + \mathcal{O}\left((F(t,\mathbf{x}) - F_0)^2\right), \tag{4}$$

where $\Omega_{(k)}(F_0,t,\mathbf{x})$ is the kth order correction to $\Omega(F(t,\mathbf{x});t,\mathbf{x})$ when $F(t,\mathbf{x})$ is expanded about the low energy or equilibrium value F_0.

It is generally known that when differential equations are linear in their dynamical function, their solutions are linear combinations of complex exponentials ([1]). Complex exponentials in turn are composed of sinusoids and/or exponential with real arguments. Thus, many systems in physics can be studied as oscillations and decays in time and space because the equations which model them are (in certain physical limits) linear.

Physical systems are studied near equilibrium or at low energy $\Longrightarrow$ Dynamical equations of motion can be linearized

$$F(t,\mathbf{x}) = F_0 + \cdots$$

$$\left[\hat{D} - \Omega_{(1)}(F_0,t,\mathbf{x})\right]F(t,\mathbf{x}) = \Omega_{(0)}(F_0,t,\mathbf{x}) + \cdots$$

$\Downarrow$

Complex exponential solutions give rise to oscillations and/or decay kinematics $\Longleftarrow$ Linear dynamical equations yield complex exponential solutions

$$F(t,\mathbf{x}) - F_0 \sim \sin(\alpha\mathbf{k}\cdot\mathbf{x} - \beta\omega t) \qquad F(t,\mathbf{x}) - F_0 \sim e^{\alpha\mathbf{k}\cdot\mathbf{x} - \beta\omega t}$$

Figure 2: Relationship between linearized dynamical equations and complex exponentials

[1]This is a paraphrase of a quote attributed to Sidney Coleman

[2]Often this equilibrium or low energy condition is the trivial statement that outside of an external perturbation, the dynamical variable is zero. In quantum mechanics, however, the defining dynamical equation is always linear in the state ket so we don't need to consider quantum systems at low energy or near equilibrium for the following argument to apply.

The explanation for the omnipresence of the quadratic-form potential (i.e., $V(F(t,\mathbf{x})) = \alpha F(t,\mathbf{x})^2/2 + \cdots$) follows a similar logic. From the Lagrangian formalism of physics, we know that a dynamical equation is linear in the dynamical variable, if and only if its associated energy is quadratic in the dynamical variable[3]. And given that the energy is a quadratic function of the dynamical variable, the potential energy (if it is non-zero) must also be a quadratic function of the dynamical variable. Thus, whenever we consider a dynamical system near equilibrium or at low energies, we should expect to find that its energy contains a "harmonic oscillator" contribution of the form $\alpha F(t,\mathbf{x})^2/2$.

Aside: Exponentials are everywhere for a different reason!

Apart from sinusoidal functions, exponential functions (with real arguments) are one of the most ubiquitous functions in physics. However, some important exponential functions in physics have origins quite different from the linear-dynamical equation origins of sinusoids. For example, in statistical physics and thermodynamics, the exponential

$$p(E) \propto e^{-\beta E}, \tag{5}$$

(where E is energy and β is inverse temperature) is foundational in studying systems at or near thermal equilibrium. Eq.(5) can be obtained by computing the probability distribution which yields the maximum uncertainty under the constraint of fixed energy. In this calculation, uncertainty is defined as[a]

$$\mathcal{S} = -\sum_i p_i \ln p_i, \tag{6}$$

where the sum is over the various energy states of the system. More interestingly, we can find other famous exponential functions like Gaussians (e.g., e^{-ax^2}) and Gamma functions (e.g., $e^{-bx}x^{c-1}$) when we maximize Eq.(**??**) under different sets of constraints. Thus, the ubiquity of exponential distributions and their generalizations throughout statistics can be seen as due to our particular representation of uncertainty as Eq.(**??**).

[a] In statistical mechanics, Botlzmann's constant k_B multiplies the right hand side, but we can imagine absorbing this constant into a redefinition of the left hand side to make the entire expression dimensionless.

- **Superposition**

  A general property of linear differential equations is that they often have multiple solutions, and the most general form of their solution is a linear combination of the individual solutions. We call this the **superposition property**. This property is physically relevant, for example, in electromagnetic systems with complicated charge and current configurations. If we have a charge distribution composed of N point charges from $k = 1, \ldots, N$, and each charge k creates an electric field $\mathbf{E}_k$, independent of the other charges, then the net electric field is $\mathbf{E}_{\text{net}} = \mathbf{E}_1 + \cdots + \mathbf{E}_N$, i.e., the sum of the electric fields of each individual charge [4]. In this example the superposition property arises from the linearity of Gauss's law, Eq.(1), but in general, superposition applies whenever a system is governed by a linear dynamical equations.

[3] This is assuming the dynamical variable has a clear interpretation in terms of energy.

[4] That physically important quantities are equivalent to the sum of the contributions of their individual parts may seem obvious but it is not generally true. For example, the total energy of a net electric field is *not* the sum of the energies of each individual field. Also, in quantum physics probabilities are nonlinear (i.e., modulus squared) functions of probability amplitudes which leads to many of the unique phase-dependent properties of quantum systems.

$$\begin{aligned} \hat{D}F_1(t,\mathbf{x}) = 0 \qquad\quad & \qquad\qquad F_3(t,\mathbf{x}) = \alpha\, F_1(t,\mathbf{x}) + \beta\, F_2(t,\mathbf{x}) \\ & \Longrightarrow \\ \hat{D}F_2(t,\mathbf{x}) = 0 \qquad\quad & \qquad\qquad \hat{D}F_3(t,\mathbf{x}) = 0 \end{aligned} \tag{7}$$

Figure 3: The superposition of solutions to homogeneous differential equations is also a solution

However, the superposition property does not always simplify calculations. An independent solution to a linear dynamical equation is termed a *mode*. If for example, the dynamical equation is a wave equation, a mode is identified by the wave number or angular frequency of the associated sinusoid. To compute physical quantities in such systems we perform a weighted sum over the contributions of each mode which can be problematic when these summations are infinite integrals.

$$\frac{E_{\text{class}}}{V} \sim \frac{\varepsilon_0}{2}\left|\int_{-\infty}^{\infty}\frac{d^3\mathbf{k}}{(2\pi)^3}\mathbf{E}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{x}}\right|^2 \sim \text{convergent}\,, \qquad \frac{E_{\text{quant}}}{V} \sim \int\frac{d^3\mathbf{k}}{(2\pi)^3}\hbar\omega_{\mathbf{k}} \sim \text{divergent} \tag{8}$$

Figure 4: In classical electrodynamics the superposition of wave solutions yields a finite energy density. In quantum electrodynamics the energy density of a photon is nominally infinite.

Indeed these summations workout just fine in the classical physics of fields and continuous media (Fig. 4), but in quantum field theory such mode summations lead to the famous divergence problems which necessitate renormalization theory.

- **Nonlinearity and perturbation**[5]

  It goes without saying that not all the fundamental equations of physics are linear, and yet we are still able to obtain useful results from them. This is because even when an equation is nonlinear we can often approximate it as linear and then solve it iteratively in terms of solutions to the linear equation.

  Thus the ubiquity of linear equations in physics is connected to the ubiquity of **perturbation theory**, the general set of techniques used to solve nonlinear equations in terms of the solutions to their linear approximations. As an example of perturbation theory, let's perturbatively solve an insoluble cubic equation[6] which becomes a soluble quadratic equation in a certain limit.

  We begin with the equation

$$\alpha x^2 - \beta = \varepsilon x^3 \tag{9}$$

  where α, β, and ε are all positive numbers. We take $\varepsilon \ll 1$, and assume that the solution to Eq.(9), can be written as an expansion in ε:

$$x = x_{(0)} + \varepsilon\, x_{(1)} + \varepsilon^2\, x_{(2)} + \cdots\,, \tag{10}$$

  where $x_{(k)}$ for $k \geq 1$ (called the **kth order correction** to the unperturbed system) is what we're trying to determine. We also assume the series Eq.(10) converges unless our solutions suggest otherwise.

  The standard approach in perturbation theory is to substitute our perturbation series *ansatz* (i.e., Eq.(10)) into the equation we're trying to solve (i.e., Eq.(9)), match terms of the same order in our perturbation parameter on both sides of the equality, then use these matchings to determine the higher order corrections. In this case, substituting the relevant equations, and matching coefficients of the same order as ε on both sides of the equality we find the system of equations

$$\alpha x_{(0)}^2 - \beta = 0$$

[5]Unlike the previous two, this last min-motif is not exclusively related to the linearity of equations of motions.

[6]Cubic equations are actually soluble, but let's assume we didn't know this.

$$
\begin{aligned}
2\varepsilon\alpha\, x_{(0)}x_{(1)} &= \varepsilon x_{(0)}^3 \\
\varepsilon^2\alpha\left(x_{(1)}^2 + 2x_{(2)}x_{(0)}\right) &= 3\varepsilon^2 x_{(0)}^2 x_{(1)} \\
&\vdots
\end{aligned} \tag{11}
$$

Solving this system up to the second-order correction we find

$$
x = \pm\left(\frac{\beta}{\alpha}\right)^{1/2} + \varepsilon\,\frac{1}{2\alpha}\left(\frac{\beta}{\alpha}\right) \pm \varepsilon^2\frac{5}{8\alpha^2}\left(\frac{\beta}{\alpha}\right)^{3/2} + \mathcal{O}(\varepsilon^3). \tag{12}
$$

This was a non-physics example, but it still bears two main features of perturbative solutions to problems in physics. First, since the higher order corrections are organized as a power series in the perturbation parameter ε, we can only expect the perturbation series to converge to a finite value if the dimensionless perturbation parameter ε is small (i.e., $\varepsilon \ll 1$). Second, each higher order correction can be expressed in terms of the unperturbed solution, $\sqrt{\beta/\alpha}$. From classical mechanics to quantum field theory, the specific ways perturbation theory is implemented changes according to what we're trying to calculate, but these features are present all perturbative calculations.

Pursuing Why

In these notes we showed that the ubiquity of oscillations, the superposition principle, and perturbation theory arises from the linearity of many fundamental equations in physics. But you could argue that we merely replaced the question of why oscillations and superpositions are ubiquitous with the question of why so many equations in physics are linear (or at least linear in a certain limit). Connected to this question is the question of what we consider to be the starting point of a theory, namely, what do we take as the premise/postulate from which all other results are derived. In modern physics, what is considered fundamental is not specifically the dynamical equation of a physical quantity, but the Lagrangian from which that dynamical equation is derived. (I won't get into the reasons for this but it has to do with how QFT is formulated). So, rephrasing the question, we are essentially asking why do the Lagrangians for so many of our physical systems, lead to linear (or approximately linear) dynamical equations, which themselves lead to the oscillation and superposition properties cited above.

Answering this question about why Lagrangians take the form they do forces us to consider another motif in physics: symmetry and simplicity.

References

[1] S. Hassani, *Mathematical physics: a modern introduction to its foundations*. Springer Science & Business Media, 2013.