Symmetry and Simplicity

These notes[1] are part of a series concerning "Motifs in Physics" in which we highlight recurrent concepts, techniques, and ways of understanding in physics. In these notes we discuss the interplay between symmetry and simplicity, in particular about how symmetries can simplify or complicate theory development.

Symmetry in Classical Physics

In physics, a student's first encounter with the notion of symmetry is often in the somewhat abstract context of Noether's theorem. However, said student would have likely made glancing contact with symmetry long before then. Frank Wilczek, who studied the symmetries defining nuclear interactions, once described a symmetry rather simply, albeit cryptically, as change without change ("Why is the world so beautiful?" [1]). More precisely, we say a physical theory has a symmetry when certain properties of the theory do not change when we change other properties. Such symmetries appear throughout introductory classical mechanics without a label identifying them as such. For example, the fact that thew equations of motion of projectile motion do not change under horizontal shifts of the system or that the equations defining the motion of planetary orbits are the same regardless of how we rotate the system around the orbital axes are both examples of symmetries.

What Noether's theorem achieves is a precise mathematical definition of these qualitative notions of a symmetry and, more interestingly, it establishes a connection between symmetry and conservation laws. Now, symmetry is an inherently mathematical property which is often first introduced in a geometry class where students study how circles and spheres do not change under rotations, or how polygons do not change under discrete transformations. And since symmetry can be formulated without reference to physical principles, there is no reason to expect the mathematical property of symmetry to be connected to anything physical. Thus, part of the wonder of what Noether achieved was to prove the connection between certain symmetries and conservation laws such that the existence of the former implies the existence of the latter. Formally, Noether's theorem states

> **Theorem 1 (Noether's Theorem)** *[2] For each continuous symmetry of the action[2] there is a conserved quantity. By "symmetry" we mean that when the coordinates of the system are changed slightly, then there is no first order change in the action. By "conserved quantity" we mean a quantity that is time-independent.*

From Noether's theorem, we find that the fact that the equations of projectile motion are independent of horizontal shifts implies that projectile motion conserves horizontal momentum, and the fact that central-force equations of motion are independent of rotations implies that central-force motion conserves angular momentum. In both of these scenarios, the conservation laws as derived from the symmetries simplify the dynamics of our system by allowing us to ignore a physical quantity which is time independent. If a physical quantity is conserved in a system, then the equations governing the quantity's dynamics reduce to

$$\frac{d}{dt}\mathcal{O}(t) = 0, \tag{1}$$

implying the trivial solution $\mathcal{O}(t) = \mathcal{O}(0)$ for all time. Hence, when a physical quantity is conserved we do not need to keep track of its time evolution and can instead focus on the less non-trivial dynamics of other parts of the system. Moreover, we can often write the more complicated equations of motion in terms of the conserved quantity, thus allowing us to parameterize the dynamics of the system by its physically

[1] Inspired by a blog post by Philip Tanedo

[2] Noether's theorem is sometimes formulated in terms of symmetries of the Lagrangian, but the Lagrangian *can* actually change under a symmetry transformation up to a first-order time derivative and the relevant system will still have a conserved quantity.

important constants of motion. It is in this way that the existence of a symmetry simplifies classical physics: continuous symmetries of the action imply conservation laws which imply that a physical observable has very simple (i.e., non-exitant) time evolution.

Symmetry in Quantum Physics

The fact that symmetries simplify the dynamics of classical systems has an analog in quantum physics[3]. First, for quantum systems we have **symmetry operators** $\hat{\mathcal{J}}$ which act on states to transform them according to the definition of the symmetry. We note that in quantum physics, the moniker symmetry is just a label for the operator and does not necessarily imply that any theory in which the symmetry operator can be defined is symmetric with respect to that operator.

What does "symmetric" mean in the context of quantum mechanics? In order to define symmetric, we need to specify some of the substructure of the symmetry operator $\hat{\mathcal{J}}$. In particular, for quantum mechanics, the symmetry operator $\hat{\mathcal{J}}$ is defined in terms of what is known as the **generator** $\hat{G}$ as

$$\hat{\mathcal{J}} = \exp\left[-\frac{i}{\hbar}\varepsilon\hat{G}\right], \tag{2}$$

where ε is the parameter corresponding to the physical quantity changed by the symmetry operation. For example, if $\hat{\mathcal{J}}$ corresponds to the symmetry operation of translation, then ε would be a parameter with units of distance. The quantity $\hat{G}$ is the operator form of the conserved quantity associated with the parameter ε. In the case where $\hat{\mathcal{J}}$ represented translation, $\hat{G}$ would be the linear-momentum operator.

Now, we say a quantum theory is "symmetric" with respect to a symmetry operator if the Hamiltonian $\hat{H}$ of the theory satisfies

$$\hat{\mathcal{J}}^{\dagger}\hat{H}\hat{\mathcal{J}} = \hat{H}, \tag{3}$$

that is, when the Hamiltonian is invariant under the symmetry operator. By Eq.(2), this definition of symmetric implies that we must also have

$$[\hat{G}, \hat{H}] = 0. \tag{4}$$

And given the Hamilton equations of motion for the expectation values of time-independent operators, Eq.(4) in turn implies

$$\frac{d}{dt}\langle G\rangle = 0 \tag{5}$$

which is to say $\langle G\rangle$, the average of the physical quantity G, is conserved in this theory. Thus the quantum physics definition of symmetric given by Eq.(3) is consistent with Noether's classical physics result that the existence of symmetries implies the existence of conserved quantities. But do such symmetries simplify, as they do in classical mechanics, our study of the dynamics of quantum systems?

We answer this question by introducing states into this picture of symmetries. Let's say we have a state $|\alpha\rangle$ which is an eigenstate of $\hat{H}$ with energy eigenvalue E_α. Under the symmetry transformation $\hat{\mathcal{J}}$, $|\alpha\rangle$ transforms to $|\alpha_J\rangle$ where

$$|\alpha_J\rangle = \hat{\mathcal{J}}|\alpha\rangle. \tag{6}$$

If our quantum theory is symmetric under $\hat{\mathcal{J}}$ (that is, if $[\hat{H}, \hat{G}] = 0$) then $|\alpha_J\rangle$ is an eigenstate of $\hat{H}$ with the energy eigenvalue E_α. We see this as follows:

$$\hat{H}|\alpha_J\rangle = \hat{H}\hat{\mathcal{J}}|\alpha\rangle = \hat{\mathcal{J}}\hat{H}|\alpha\rangle = E_\alpha|\alpha_J\rangle, \tag{7}$$

Thus, E_α is a degenerate eigenvalue of $\hat{H}$ and is the eigenvalue of the eigenstate $|\alpha\rangle$, in addition to being the eigenvalue of all independent transformations of $|\alpha\rangle$. Moreover, given $[\hat{H}, \hat{G}] = 0$, we know that $|\alpha\rangle$ is an eigenstate of both $\hat{G}$ as and $\hat{H}$. Thus, if we have a set of quantum numbers ℓ such that the eigenvalues of $\hat{G}$

[3]The following section is adapted from [3].

are g_ℓ with

$$\hat{G}|\alpha, \ell\rangle = g_\ell|\alpha, \ell\rangle, \tag{8}$$

where we redefined $|\alpha\rangle \to |\alpha, \ell\rangle$ in order to represent quantum numbers associated with both $\hat{H}$ and $\hat{G}$. Then, by Eq.(7) we find

$$\hat{H}|\alpha, \ell\rangle = E_\alpha|\alpha, \ell\rangle. \tag{9}$$

which shows that the quantum number ℓ associated with the conserved operator $\hat{G}$ does not parameterize the energy spectrum of the theory. This fact is often not taken to be as important as Noether's theorem, but for narrative symmetry we write it as the theorem it is.

Theorem 2 *For each continuous symmetry of the Hamiltonian, there is a degeneracy in the energy spectrum. By "symmetry" we mean that Eq.*(3) *is satisfied. By "degeneracy" we mean multiple linearly-independent eigenstates have the same energy eigenvalue.*

The standard example of this result is in the quantum physics of central potentials. When a particle is subjected to a central potential, the particle's Hamiltonian is invariant under rotation transformations. Consequently, the Hamiltonian commutes with the angular momentum operator (i.e., the generator of rotations) and the energy eigenvalues of the Hamiltonian are degenerate with respect to the "magnetic quantum number" m, the quantum number defining the eigenvalues of the angular momentum operator[4].

By virtue of the Schrödinger equation, the dynamics of quantum systems are defined by the Hamiltonian operator which is itself defined by its energy eigenspectrum. Therefore, the fact that continuous symmetries in quantum physics lead to energy degeneracies implies that fewer quantum numbers are needed to characterize the independent time-evolving phases of states. This in turn simplifies our study of the dynamics of quantum systems.

In summary, while continuous symmetries simplify classical physics by giving physical observables trivial dynamics, continuous symmetries simplify quantum systems by reducing the number of parameters (i.e., quantum numbers) which define the energy eigenspectrum.

Symmetries and High-Energy Physics

Outside of simplifying the dynamics of physical systems, the knowledge that a system obeys certain symmetries often streamlines the process of finding physical models for the system. For example, if we know a system conserves angular momentum, then, according to Noether's theorem, we also know that the system's Lagrangian cannot include any explicit angular variables. Thus, when searching for possible Lagrangians for the system, we can ignore any Lagrangians which include angular variables. This procedure is general for all physical theories: Knowing a system obeys a certain symmetry narrows the search for possible models of the system.

But today, high-energy physics (i.e., the 21st century label for particle physics) has a complicated relationship with symmetry. On one level, symmetry in high-energy physics operators as it always has in other fields, that is it serves as a means of constraining the possible space of theories and making the analysis of those theories simpler. But assuming the existence of a never before observed symmetry has the corollary of introducing physical effects required to maintain the consistency of the symmetry. Thus although symmetries can (and often do) make theory construction simpler by constraining the space of possible theories according to some principle, symmetries can also complicate our theories by positing the existence of phenomena never before seen (See Fig. 1).

[4]Even though central potential systems have the property $d\langle \mathbf{L}^2\rangle/dt = 0$, their energy eiegnvalues can still be labeled by ℓ, the quantum number of the $\hat{\mathbf{L}}^2$ operator, because $\hat{\mathbf{L}}^2$ is not associated with a symmetry transformation and thus Theorem 2 does not apply.

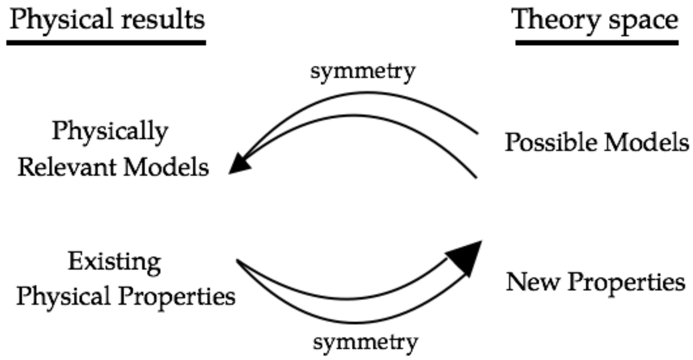

Figure 1: Symmetry's complicated relationship with theory construction. Symmetries can narrow the space of possible theories, but imposing new symmetries on physical systems often requires postulating new physical phenomena or properties of nature.

This is essentially how the **standard model** of particle physics was developed. For each fundamental interaction there is a symmetry which constrains how the interaction manifests in the physical world. Through the postulation of such symmetries, physicists were able to narrow down the possible theories which described nuclear, radioactive, and electromagnetic interactions. But these theories also required, if the symmetries were true to nature, new particles which had never been observed. Hence from these symmetries physicists predicted the existence of the W^+ and W^- bosons, the Z^0 boson, and the Higgs boson years before any of these particles were observed in experiments.

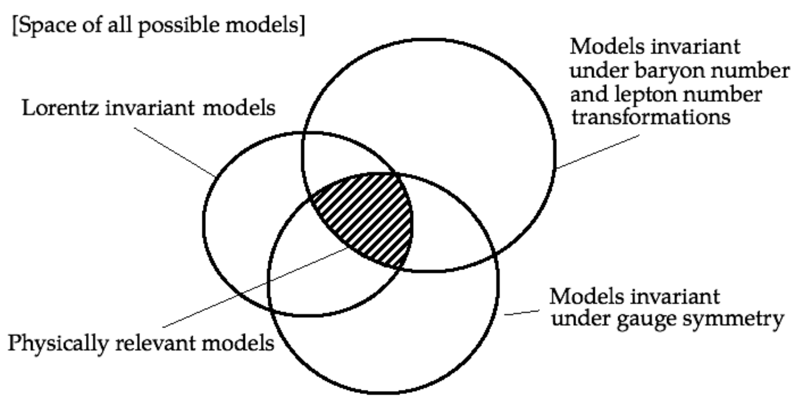

Figure 2: Schematic representation of the way symmetries constrain the possible theories of the standard model. The sizes of the ovals do not reflect the actual "size" of the space of possible models for each symmetry

The story of the standard model and symmetry represents an ideal interplay between symmetry and theory construction, and, following its example, physicists have tried to repeat the story in developing theories beyond the standard model. The most popular class of such theories are **supersymmetric** theories. Supersymmetry is a symmetry which transforms bosons into fermions and vice versa, very much analogous to the way rotational symmetry transforms constant-length vectors into other constant-length vectors. Supersymmetry was originally postulated in order to eliminate divergent results in quantum field theories, but much like the symmetries of the standard model, it also came with some baggage. In order for a theory to be supersymmetric, for every boson in the theory there must be a fermion of the same mass and similar interaction properties. So in order to incorporate supersymmetry into the standard model, physicists had to effectively double the number of particles in the universe. Moreover, given that it is clear that mass-degenerate boson-fermion pairs have never been observed, physicists had to further postulate that if supersymmetry exists, it must be a **broken symmetry**, meaning it is a symmetry of the mathematical formalism, but said symmetry does not appear in the physical solutions of the formalism[5].

You may think this is an ass-backwards method of theory construction—introducing a symmetry to solve one physical problem and then partially eliminating the symmetry to solve the problems created by its introduction—and many would agree with you, but physicists, like any community, are inevitably influenced by their history, and so it is not terribly surprising that an approach to developing physical theories which was found to be so successful in the past (e.g., in the standard model) would be continuously applied until exhaustion, or until a novel approach to theory development is found.

Why symmetry?

I've discussed the utility of symmetry, but I have not really touched upon why the universe bears the symmetries it does. Why is Lorentz invariance fundamental? Why are fundamental interactions governed by special symmetries? Alternatively, with Noether's theorem we can ask the complementary question: Why are certain physical quantities conserved? It is not clear to me which question is the more fundamental one, and it is even less clear what either of their answers are. By and large symmetries seem to be aesthetic properties which are valued based on particularly human conceptions of order and logical unity. And, as far as I can tell, it is not clear why the physical universe, which is by most rational accounts indifferent to human machinations, would bear an of its current fidelity to them.

References

[1] F. Wilzcek, "Why is the world so beautiful?," April 2016.

[2] D. Morin, *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.

[3] J. J. Sakurai, S.-F. Tuan, and E. D. Commins, *Modern quantum mechanics, revised edition*. AAPT, 1995.

[5]The classic example of this is a ferromagnet. Maxwell's equations are rotationally symmetric, but we say ferromagnets "break" rotational symmetry because their magnetic fields are not rotationally symmetric.