Physics 143a – Workshop 2

On Rotations, Commutations, and Change of Bases

Week Summary

- **Rotation Operator:** The rotation operator is a unitary operator which can be applied to states and other operators to transform them into their results in a rotated coordinate system. For example, the operator for rotation by ϕ around the $\mathbf{k}$ (i.e., + z) direction is

$$\hat{R}(\phi\mathbf{k}) = \exp\left(-\frac{i}{\hbar}\hat{J}_z\phi\right) \tag{1}$$

  where $\hat{J}_z$ is the hermitian z-direction angular momentum operator.
  Rotation operators about different directions do not commute (e.g., $\hat{R}(\phi_1\mathbf{j})\hat{R}(\phi_2\mathbf{k}) \neq \hat{R}(\phi_2\mathbf{k})\hat{R}(\phi_1\mathbf{j})$), a fact related to the non-commutation of angular momentum operators:

$$[\hat{J}_k, \hat{J}_\ell] = i\hbar\,\epsilon_{k\ell m}\hat{J}_m \tag{2}$$

  where $\epsilon_{k\ell m}$ is +1 if (k, ℓ, m) are even permutations of $(1, 2, 3)$, -1 if (k, ℓ, m) are odd permutations of $(1, 2, 3)$, and 0 if any (k, ℓ, m) has a repeated component.

- **Commuting Observables:** Observables represented by the operators $\hat{A}$ and $\hat{B}$ can be measured simultaneously (i.e., each measured without affecting the measurement of the other) if the operators commute. This result is summarized by the uncertainty principle

$$\Delta_\phi\hat{A}\Delta_\phi\hat{B} \geq \frac{1}{2}\left|\langle\phi|[\hat{A},\hat{B}]|\phi\rangle\right|, \tag{3}$$

  where $\Delta_\phi\hat{O} = \sqrt{\langle\phi|\hat{O}^2|\phi\rangle - \langle\phi|\hat{O}|\phi\rangle^2}$. A standard example is angular momentum (orbital and spin), with the angular momentum for various direction unable to be measured without affecting the angular momentum measurement for other directions (due to the non-commutation expressed by Eq.(2))

- **Change of Basis:** States and operators when written in vector and matrix notation, respectively, are always written in some eigenbasis (say $|\alpha_i\rangle$ for $i = 1, \ldots, N$) of orthonormal eigenvectors which span the state space. To transform the states and operators to a new basis, say $|\beta_i\rangle$, we apply the transformations[1]

$$\left[\hat{A}\right]_{\beta_i\text{ basis}} = \hat{U}\left[\hat{A}\right]_{\alpha_i\text{ basis}}\hat{U}^\dagger, \tag{4}$$

  and

$$\left[|\psi\rangle\right]_{\beta_i\text{ basis}} = \hat{U}\left[|\psi\rangle\right]_{\alpha_i\text{ basis}} \tag{5}$$

  where the elements of the unitary matrix $\hat{U}$ (which transforms from the α basis *to* the β basis) are defined by $\hat{U}_{ij} = \langle\beta_i|\alpha_j\rangle$. Generally, the change of basis matrix $\hat{U}$ is defined as

$$\hat{U}_{ij} = \langle\text{final basis}_i|\text{initial basis}_j\rangle. \tag{6}$$

[1] We note Townsend uses $\mathbb{S}^\dagger\hat{A}\mathbb{S}$ to denote this transformation, and thus the $\hat{U}$ defined in Eq.(6) is $\hat{U} = \mathbb{S}^\dagger$. Regardless of the definition of the unitary operator, what is important is that the "final basis" appears as a bra on the left hand side of the operator/state.

1 Problems

1. **Commutators and degenerate eigenvalues** [From Last Week]
   Let us take $N \times N$ matrices A, B, and C satisfying
   $$[A, B] = 0, \quad [A, C] = 0, \quad [B, C] \neq 0. \tag{7}$$
   Let us assume that A has a non-degenerate space eigenvalues. Is this possible?[2]

2. **Rotations don't commute! Or do they?**
   Alice and Betty are having an argument about rotation in three dimensions. Alice believes that rotations (classical and quantum) about orthogonal axes do not commute, while Betty thinks there are exceptions to this claim.

   Betty: I'm telling you, it's possible to have $[\hat{J}_i, \hat{J}_j] = 0$ when $i \neq j$.

   Alice: Nonsense!

   Betty: No, listen. If I rotate a system by an auxiliary angle ϕ and then by an azimuthal angle[3] θ, then I would get the same resulting system as if I performed the azimuthal θ rotation *before* the auxiliary ϕ rotation. These rotations are in orthogonal directions but their order doesn't matter. They commute!

   Who is right and why?

3. **Practicing Spins**
   Suppose our particle is in the state
   $$|\chi\rangle = \frac{1+i}{\sqrt{6}}|+, \mathbf{z}\rangle + \frac{2}{\sqrt{6}}|-, \mathbf{z}\rangle. \tag{8}$$
   (a) What are the probabilities of getting $+\hbar/2$ and $-\hbar/2$ if you measure S_z? What is $\langle S_z \rangle$?
   (b) What are the probabilities of getting $+\hbar/2$ and $-\hbar/2$ if you measure S_x? (Note: this concerns a different measurement for the state $|\chi\rangle$, *not* another measurement which occurs after that in (b).) What is $\langle S_x \rangle$?

4. **Sequential Measurements**
   An operator $\hat{A}$, representing observable A, has two normalized eigenstates $|\alpha_1\rangle$ and $|\alpha_2\rangle$, with eigenvalues a_1 and a_2, respectively. Operator $\hat{B}$, representing observable B, has two normalized eigenstates $|\beta_1\rangle$ and $|\beta_2\rangle$, with eigenvalues b_1 and b_2. The two sets of eigenstates are related by
   $$|\alpha_1\rangle = (3|\beta_1\rangle + 4|\beta_2\rangle)/5, \quad |\alpha_2\rangle = (4|\beta_1\rangle - 3|\beta_2\rangle)/5. \tag{9}$$
   (a) Observable A is measured, and the value a_1 is obtained. What is the state of the system immediately after this measurement?
   (b) If B is now measured, what are the possible results, and what are their probabilities?
   (c) Right after the measurement of B, A is measured again. What is the probability of getting a_1? (Note: the answer would be different if I had told you the outcome of the B measurement)
   (d) Express $[\hat{A}, \hat{B}]$ in the $\{|\alpha_1\rangle, |\alpha_2\rangle\}$ basis. Interpret this result in the context of Eq.(3) and the results of the previous parts.

[2] Last week, this problem was phrased as: *Show that at least one eigenvalue of A is degenerate. Why is $[B, C] \neq 0$ important in establishing this?*

[3] These angles are the ones which define the arbitrary spherical coordinate direction $\mathbf{n} = \sin\theta\cos\phi\mathbf{x} + \sin\theta\sin\phi\mathbf{y} + \cos\theta\mathbf{z}$.

2 Solutions

1. (On previous week's solution set)

2. It is not possible to have $[J_i, J_j] = 0$ when $i \neq j$. Betty is confusing two things: rotation operators and the generators which define those operators. A general quantum mechanical rotation operator by an angle ϕ about a direction $\vec{n}$ is defined as

$$\hat{R}(\vec{n}, \phi) = \exp\left(-i\phi \frac{\vec{n} \cdot \vec{\hat{J}}}{\hbar}\right). \tag{10}$$

where $\vec{\hat{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$ defines the *generators* of rotations in $\mathbb{R}^3$. By definition, these generators *must* satisfy

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hbar\hat{J}_k. \tag{11}$$

where ϵ_{ijk} is the Levi civita symbol. When Betty claims that rotations can commute she is implicitly talking about Eq.(10), but she then incorrectly references the commutators of the generators of those rotations.

This is not to say that rotations operators actually do commute. In this regard, Betty is confusing angles which define the coordinates of an object and angles which define a rotation operation on a coordinate system.

For example if we were to place a heavy top at an angle $\theta = \theta'$ and $\phi = 0$ with respect to the z axis and then rotate that top about the z axis by an angle $\phi = \phi'$, then the heavy top would be in the same configuration if we were to first rotate it by an angle $\phi = \phi'$ about the z axis and then place it at an angle $\theta = \theta'$ with respect to the z axis. However, these two "rotations" do not correspond to the Rotation operators for the $\mathbb{R}^3$ coordinate system; they instead define coordinates on a sphere in $\mathbb{R}^3$. It is specifically, the rotation operators[4] which do not commute.

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3. (a) For the state

$$|\chi\rangle = \frac{1+i}{\sqrt{6}}|+, \mathbf{z}\rangle + \frac{2}{\sqrt{6}}|-, \mathbf{z}\rangle, \tag{12}$$

the probability to get a $+\hbar/2$ value upon measurement of S_z is the modulus squared of the coefficient of the $|+, \mathbf{z}\rangle$ state. Thus we have

$$\text{Prob}(S_z = +\hbar/2) = \left|\frac{1+i}{\sqrt{6}}\right|^2 = \frac{1}{3}. \tag{13}$$

Similarly, we have the probability to get a $-\hbar/2$ value upon measurement of S_z is

$$\text{Prob}(S_z = -\hbar/2) = \left|\frac{2}{\sqrt{6}}\right|^2 = \frac{2}{3}. \tag{14}$$

Thus the average value of S_z is

$$\langle S_z \rangle = \frac{\hbar}{2}\frac{1}{3} + \left(-\frac{\hbar}{2}\right)\frac{2}{3} = -\frac{\hbar}{6}. \tag{15}$$

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[4]In more mathematical parlance, these operators are the generators defining the $\mathbb{R}^3$ rotation group of SO(3).

(b) To find the corresponding results for S_z, it is easiest to express $|\chi\rangle$ in the $|\pm,\mathbf{x}\rangle$ basis. To do so we use the basis relations

$$|+,\mathbf{z}\rangle = \frac{1}{\sqrt{2}}|+,\mathbf{z}\rangle + \frac{1}{\sqrt{2}}|+,\mathbf{z}\rangle, \tag{16}$$

$$|-,\mathbf{z}\rangle = \frac{1}{\sqrt{2}}|+,\mathbf{x}\rangle - \frac{1}{\sqrt{2}}|+,\mathbf{x}\rangle. \tag{17}$$

Thus $|\chi\rangle$ in the $|\pm,\mathbf{x}\rangle$ basis is

$$|\chi\rangle = \frac{3+i}{\sqrt{12}}|+,\mathbf{x}\rangle + \frac{-1+i}{\sqrt{12}}|-,\mathbf{x}\rangle. \tag{18}$$

The probability to get a $+\hbar/2$ value upon measurement of S_x is then

$$\mathrm{Prob}(S_x = +\hbar/2) = \left|\frac{3+i}{\sqrt{12}}\right|^2 = \frac{5}{6}. \tag{19}$$

And the average value of S_x is

$$\langle S_x\rangle = \frac{\hbar}{2}\frac{5}{6} + \left(-\frac{\hbar}{2}\right)\frac{1}{6} = \frac{\hbar}{3}. \tag{20}$$

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4. (a) By one of the postulates of quantum mechanics, measurements of observables only yield eigenvalues of the observable's operator. And subsequent to the measurement, the system is in the eigenket corresponding to the measured eigenvalue. Thus after obtaining a measurement of a_1, the system must be in the state $|\alpha_1\rangle$.

(b) If the system is in the state $|\alpha_1\rangle$, then, upon measurement of B, the probability of obtaining the possible eigenvalues of B are

$$\mathrm{Prob}(B = b_1) = |\langle\beta_1|\alpha_1\rangle|^2 = \frac{9}{25} \tag{21}$$

$$\mathrm{Prob}(B = b_2) = |\langle\beta_2|\alpha_1\rangle|^2 = \frac{16}{25}. \tag{22}$$

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(c) The probability to obtain the measurement a_1 after passing through an apparatus that measures the value of B is equal to the sum of the conditional probabilities of each possible result of B given $A = a_1$, weighted by the probability of the result B. Mathematically,

$$\mathrm{Prob}_{\text{final}}(a_1) = \mathrm{Prob}(a_1|b_1)\mathrm{Prob}_{\text{initial}}(b_1) + \mathrm{Prob}(a_1|b_2)\mathrm{Prob}_{\text{initial}}(b_2). \tag{23}$$

The quantities $P_{\text{initial}}(b_{1,2})$ are what we found in (a). The conditional probabilities $P(a_1|b_{1,2})$ are the probabilities of obtaining a measurement of a_1 given that the system is in the state $|\beta_{1,2}\rangle$. By the mathematical properties of the modulus, these conditional probabilities are also identical to what we found in (a), but for explicitness we calculate them explicitly.

Inverting the $|\alpha_{1,2}\rangle$-$|\beta_{1,2}\rangle$ basis equations we find

$$|\beta_1\rangle = (3|\alpha_1\rangle + 4|\alpha_2\rangle)/5, \quad |\beta_2\rangle = (4|\alpha_1\rangle - 3|\alpha_2\rangle)/5. \tag{24}$$

Thus the conditional probabilities to obtain $A = a_1$ when one is in the state $|\beta_1\rangle$ or $|\beta_2\rangle$ are, respectively,

$$\text{Prob}(a_1|b_1) = |\langle\alpha_1|\beta_1\rangle|^2 = \frac{9}{25} \tag{25}$$

$$\text{Prob}(a_1|b_2) = |\langle\alpha_1|\beta_2\rangle|^2 = \frac{16}{25}. \tag{26}$$

Thus, the probability to obtain a measurement of a_1 upon measuring A (after passing through an apparatus which measures B with some unknown result) is

$$\text{Prob}_{\text{final}}(a_1) = \frac{9^2}{25^2} + \frac{16^2}{25^2} = \frac{337}{625}. \tag{27}$$

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(d) To compute $[\hat{A}, \hat{B}]$ in the basis $|\alpha_{1,2}\rangle$ we need to express $\hat{A}$ and $\hat{B}$ in the appropriate basis. In its own basis, $\hat{A}$ has exclusively diagonal elements defined by its eigenvalues

$$\hat{A} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad \text{[In the } |\alpha_{1,2}\rangle \text{ basis]}. \tag{28}$$

$\hat{B}$ has a similar form in its own basis:

$$\hat{B} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \qquad \text{[In the } |\beta_{1,2}\rangle \text{ basis]}. \tag{29}$$

To find $\hat{B}$ in the $|\alpha_{1,2}\rangle$ basis we use the change of basis matrix implied by Eq.(9). Namely, the matrix to change from the $|\beta_{1,2}\rangle$ basis to the $|\alpha_{1,2}\rangle$ basis is

$$U = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \qquad \text{[To go from } |\beta_{1,2}\rangle \text{ to } |\alpha_{1,2}\rangle \text{ basis]}. \tag{30}$$

Thus, $\hat{B}$ in the desired basis is

$$\begin{aligned} \hat{B} &= U\hat{B}U^\dagger \\ &= \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \\ &= \frac{1}{25}\begin{pmatrix} 9b_1 + 16b_2 & 12(b_1 - b_2) \\ 12(b_1 - b_2) & 16b_1 + 9b_2 \end{pmatrix} \qquad \text{[In the } |\alpha_{1,2}\rangle \text{ basis]}. \end{aligned} \tag{31}$$

We thus find that the commutator of $\hat{A}$ and $\hat{B}$ in the $|\alpha_{1,2}\rangle$ basis is

$$\begin{aligned} [\hat{A}, \hat{B}] &= \frac{1}{25}\left[\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} 9b_1 + 16b_2 & 12(b_1 - b_2) \\ 12(b_1 - b_2) & 16b_1 + 9b_2 \end{pmatrix}\right] \\ &= \frac{12}{25}(b_1 - b_2)(a_1 - a_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{[In the } |\beta_{1,2}\rangle \text{ basis]}, \end{aligned} \tag{32}$$

which is not equal to zero. We note that although the precise form of $[\hat{A}, \hat{B}]$ is dependent on the basis in which we express it, the fact that it is nonzero is basis independent. By the Uncertainty principle

$$\Delta_\phi \hat{A} \Delta_\phi \hat{B} \geq \frac{1}{2}\left|\langle\phi|[\hat{A}, \hat{B}]|\phi\rangle\right|, \tag{33}$$

the fact that $[\hat{A}, \hat{B}] \neq 0$ implies that it is not possible to precisely measure A and B simultaneously.

Interpreted differently, a relatively precise measurement of A would require a relatively imprecise measurement of B.

We should note that Eq.(33) is not clearly meaningful in this problem, because since we are dealing with an eigenket of $\hat{A}$ in the initial part of the problem, $\Delta_\phi A = 0$ and $\langle\phi|[\hat{A}, \hat{B}]|\phi\rangle = 0$ because $|\phi\rangle = |\alpha_1\rangle$.