Physics 143a – Workshop 8

Coherent States and Quantum Mechanics Miscellany

Week Summary

- **Coherent State:** Coherent linear superpositions of harmonic oscillator eigenstates (known simply as "coherent states") are special states which minimize the Heisenberg uncertainty relation for harmonic oscillators. They are eigenstates of the lowering operator, i.e.,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{1}$$

and they are written in terms of energy eigenstates $|n\rangle$ as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2}$$

For the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2, \tag{3}$$

coherent states satisfy

$$\langle\alpha|\hat{X}|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\,\alpha \tag{4}$$

$$\langle\alpha|\hat{P}|\alpha\rangle = \sqrt{2\hbar m\omega}\,\mathrm{Im}\,\alpha \tag{5}$$

$$\langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right). \tag{6}$$

Computing the uncertainty in the Hamiltonian $\hat{H}$ for a coherent state, we find $\Delta_\alpha\hat{H} = \hbar\omega|\alpha|$. Given Eq.(6) this uncertainty implies

$$\frac{\Delta_\alpha\hat{H}}{\langle H\rangle} \sim \frac{1}{|\alpha|} \to 0 \qquad \text{if } |\alpha| \to \infty. \tag{7}$$

Because the existence of fundamental uncertainty in a measurement is a quantum mechanical attribute, the result Eq.(7) allows us to interpret the $|\alpha| \to \infty$ limit of the coherent state as an approach toward classical behavior.

1 Problems

1. **Hellmann-Feynman Theorem**

   (a) Let a Hamiltonian $\hat{H}$ depend on a parameter λ: $\hat{H} = \hat{H}(\lambda)$. Let $E(\lambda)$ be a nondegenerate eigenvalue and $|\varphi(\lambda)\rangle$ be the corresponding normalized eigenvector $\langle\varphi(\lambda)|\varphi(\lambda)\rangle = 1$:

   $$\hat{H}(\lambda)|\varphi(\lambda)\rangle = E(\lambda)|\varphi(\lambda)\rangle. \tag{8}$$

   Prove the Feynman-Hellman theorem[1]:

   $$\frac{\partial E}{\partial \lambda} = \left\langle \varphi(\lambda)|\frac{\partial \hat{H}}{\partial \lambda}|\varphi(\lambda)\right\rangle. \tag{9}$$

   (b) The Hamiltonian for a harmonic oscillator is

   $$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2. \tag{10}$$

   i. Using Eq.(9), compute

   $$\langle n|\hat{X}^2|n\rangle, \tag{11}$$

   (where $|n\rangle$ is an energy eigenstate of the harmonic oscillator.

   ii. Using the result from i. and Eq.(9), compute

   $$\langle n|\hat{P}^2|n\rangle. \tag{12}$$

   iii. Noting that $\langle n|\hat{X}|n\rangle = 0$ and $\langle n|\hat{P}|n\rangle = 0$, compute the uncertainty product $\Delta\hat{X}\Delta\hat{P}$ for energy eigenstates of the harmonic oscillator.

   iv. If a classical system corresponds to one which is far from the uncertainty limit, what condition defines whether we consider our harmonic oscillator system to be classical?

[1] Hellman proved the theorem in 1937 [1], a year before he was executed in the Great Purge. Feynman proved this theorem as part of his undergraduate thesis at MIT [2] .

2 Solutions

1. (a) To prove the result we begin with the definition of the eigenvalue $E(\lambda)$ for an eigenstate $|\varphi(\lambda)\rangle$:

$$E(\lambda) = \langle\varphi(\lambda)|\hat{H}(\lambda)|\varphi(\lambda)\rangle. \tag{13}$$

Differentiating this result with respect to λ, we find

$$\begin{aligned}\frac{\partial E}{\partial \lambda} &= \frac{\partial}{\partial \lambda}\Big[\langle\varphi(\lambda)|\hat{H}(\lambda)|\varphi(\lambda)\rangle\Big] \\ &= \Big[\frac{\partial}{\partial \lambda}\langle\varphi(\lambda)|\Big]\hat{H}(\lambda)|\varphi(\lambda)\rangle + \left\langle \varphi(\lambda)|\frac{\partial \hat{H}}{\partial \lambda}|\varphi(\lambda)\right\rangle + \langle\varphi(\lambda)|\hat{H}(\lambda)\Big[\frac{\partial}{\partial \lambda}|\varphi(\lambda)\rangle\Big] \\ &= E(\lambda)\Big[\frac{\partial}{\partial \lambda}\langle\varphi(\lambda)|\Big]|\varphi(\lambda)\rangle + \left\langle \varphi(\lambda)|\frac{\partial \hat{H}}{\partial \lambda}|\varphi(\lambda)\right\rangle + E(\lambda)\langle\varphi(\lambda)|\Big[\frac{\partial}{\partial \lambda}|\varphi(\lambda)\rangle\Big] \\ &= E(\lambda)\frac{\partial}{\partial \lambda}\Big[\langle\varphi(\lambda)|\varphi(\lambda)\rangle\Big] + \left\langle \varphi(\lambda)|\frac{\partial \hat{H}}{\partial \lambda}|\varphi(\lambda)\right\rangle \\ &= \left\langle \varphi(\lambda)|\frac{\partial \hat{H}}{\partial \lambda}|\varphi(\lambda)\right\rangle,\end{aligned} \tag{14}$$

where we used the fact that the state is normalized in the final line. ■

(b) i. For the harmonic oscillator, we have the eigenvalues and Hamiltonian

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad \hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega^2\hat{X}^2. \tag{15}$$

Taking ω to be the relevant parameter in the Feynman-Hellmann theorem, we have

$$\begin{aligned}\frac{\partial E_n}{\partial \omega} &= \langle n|\frac{\partial \hat{H}}{\partial \omega}|n\rangle \\ \hbar\left(n + \frac{1}{2}\right) &= m\omega\langle n|\hat{X}^2|n\rangle,\end{aligned} \tag{16}$$

thus we find

$$\langle n|\hat{X}^2|n\rangle = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right). \tag{17}$$

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ii. Taking m to be the relevant parameter in the Feynman-Hellmann theorem, we have

$$\begin{aligned}\frac{\partial E_n}{\partial m} &= \langle n|\frac{\partial \hat{H}}{\partial m}|n\rangle \\ 0 &= -\frac{1}{2m^2}\langle n|\hat{P}^2|n\rangle + \frac{1}{2}\omega^2\langle n|\hat{X}^2|n\rangle.\end{aligned} \tag{18}$$

Therefore the expectation value of the momentum-squared operator is

$$\langle n|\hat{P}^2|n\rangle = m^2\omega^2\langle n|\hat{X}^2|n\rangle = \hbar m\omega\left(n + \frac{1}{2}\right). \tag{19}$$

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iii. Given that $\langle n|\hat{X}|n\rangle = 0$ and $\langle n|\hat{P}|n\rangle = 0$, we find that the uncertainty limit for a harmonic oscillator eigenstate is

$$\Delta\hat{X}\Delta\hat{P} = \sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}\sqrt{\hbar m\omega\left(n+\frac{1}{2}\right)} = \hbar\left(n+\frac{1}{2}\right). \tag{20}$$

iv. The uncertainty limit is $\Delta\hat{X}\Delta\hat{P} = \hbar/2$. Thus the result Eq.(20) would be far from the uncertainty limit (and would approximate classical behavior) if $n \gg 1$.

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References

[1] H. Hellmann, "Einfuhrung in dei quanten-chemie," 1937.

[2] R. P. Feynman, "Forces in molecules," *Physical Review*, vol. 56, no. 4, p. 340, 1939.