Section Problems: Week of July 27

1. **Lincoln Continental**

   A Lincoln Continental is twice as long as a VW Beetle, when they are at rest. As the continental overtakes the VW, going through a speed trap, a (stationary) policeman observes that they both have the same length. The VW is going at half the speed of light. How fast is the Lincoln going? Leave your answer as a multiple of c.

   Solution:

   We know that the Lincoln Continental is twice as long as the VW Beetle when they are at rest ($L_C = 2L_{VW}$). We also know that from perspective of someone on the ground they have the same length ($\overline{L}_C = \overline{L}_{VW}$) when they are moving with speed v_{LC} and v_{VW} respectively. Lastly, we know that $v_{VW} = c/2$. Using this information we can solve for the velocity of the Lincoln Continental v_{LC}. We have

$$\begin{aligned}
\overline{L}_C &= \overline{L}_{VW} \\
\frac{L_{LC}}{\gamma_{LC}} &= \frac{L_{VW}}{\gamma_{VW}} \\
\frac{2L_{VW}}{\gamma_{LC}} &= \frac{L_{VW}}{\gamma_{VW}} \\
\frac{2}{\gamma_{LC}} &= \frac{1}{\gamma_{VW}} \\
2\sqrt{1-\frac{v_{LC}^2}{c^2}} &= \sqrt{1-\frac{v_{VW}^2}{c^2}} \\
4\left(1-\frac{v_{LC}^2}{c^2}\right) &= (1-14) \\
\left(1-\frac{v_{LC}^2}{c^2}\right) &= \frac{3}{16} \\
\frac{v_{LC}^2}{c^2} &= 1-\frac{3}{16} = \frac{13}{16}.
\end{aligned} \tag{1}$$

   Taking the positive root, we find

$$v_{LC} = c\frac{\sqrt{13}}{4}. \tag{2}$$

   ■.

2. **Simpson's Paradox**

A school has 6 departments. Below are the number of applicants of men and women who applied to each department and the percentage admitted.

Department	Men		Women	
	Applicants	Admitted	Applicants	Admitted
A	825	62%	108	**82%**
B	560	63%	25	**68%**
C	325	**37%**	593	34%
D	417	33%	375	**35%**
E	191	**28%**	393	24%
F	373	6%	341	**7%**

Figure 1

From the data, most departments admit a higher percentage of women applicants than men applicants. What percent of all men applicants were admitted? What percent of all women applicants were admitted? What is interesting about this result?

Solution:

The students find the total number of men applicants and the total number of men applicants admitted, and use that to calculate the percentage of men applicants that is admitted. The same procedure is applied for women:

	Applicants	Admitted
Men	2274	**52.6%**
Women	1835	30.3%

Figure 2

This switching of trends between data when seen in smaller groups versus when the data is amalgamated is known as Simpson's paradox.

■

3. **Relative Velocity**

A and B travel at $4c/5$ and $3c/5$, respectively, with respect to the ground, as shown in Fig. 2. How fast should C travel so that she sees A and B approaching her at the same speed? What is this speed?

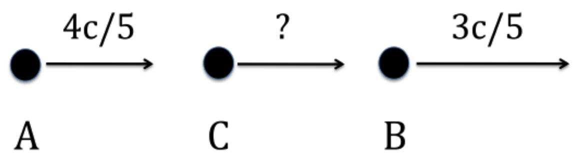

Figure 3: Relative Velocity

Solution:

Qualitatively, we should note that within the frame of C, A is moving toward C (from the left) at a certain speed and B is moving toward C (from the right) at that same speed but in the opposite direction.

Let us label the velocity of A relative to C as $+u$. Then according to the problem statement, the velocity of B relative to C must be $-u$. Using the relativistic transformation formula for velocity, we then have

$$+u = \frac{v_C - v_A}{1 - v_C v_A/c^2} \tag{3}$$

$$-u = \frac{v_C - v_B}{1 - v_C v_B/c^2}, \tag{4}$$

where we defined v_B and v_A to be positive quantities and the letters label the ground-frame velocities of the corresponding particle. From the problem statement, we know that $v_A = 4c/5$ and $v_B = 3c/5$. Solving for v_C given Eq.(3) and Eq.(4) we find

$$\begin{aligned}
u &= u \\
\frac{v_C - v_A}{1 - v_C v_A/c^2} &= \frac{v_B - v_C}{1 - v_C v_B/c^2} \\
\frac{v_C - 4c/5}{1 - 4v_C/5c} &= \frac{3c/5 - v_C}{1 - 3v_C/5c} \\
\left(v_c - \frac{4c}{5}\right)\left(1 - \frac{3v_c}{5c}\right) &= \left(\frac{3c}{5} - v_C\right)\left(1 - \frac{4v_c}{5c}\right) \\
v_C - \frac{4c}{5} + \frac{12v_c}{25} - \frac{3v_c^2}{c} &= -v_C + \frac{4v_c^2}{5c} - \frac{12v_C}{25} + \frac{3c}{5} \\
0 &= -2v_C + \frac{7v_C^2}{5c} - \frac{24v_c}{25} + \frac{7c}{5} \\
&= \frac{7v_c^2}{5c} - \frac{74v_C}{25} + \frac{7c}{5} \\
&= (35v_C^2 - 74cv_c + 35c^2)/25c \\
&= (5v_C - 7c)(7v_c - 5c)
\end{aligned} \tag{5}$$

The final line produces two solutions: $v_C = 7c/5$ and $v_C = 5c/7$. But since the speed of an object can never exceed the speed of light, the first solution is extraneous and we have

$$v_C = 5c/7. \tag{6}$$

■

4. **Proofreading**

Two people each proofread the same book. One person finds 100 errors and the other finds 60. There are 20 errors common to both people. Assume that all the errors are equally likely to be found, and also that the discovery of that error by one person is independent of the discovery of that error by the other person. Given these assumptions roughly how many errors does the book have? How many errors did both people miss?

Solution: (transcribed from Morin's *Probability for the Enthusiastic Beginner*)

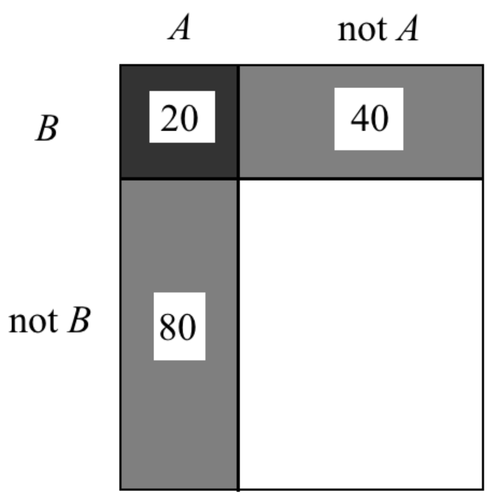

Figure 4: Figure not precisely to scale

The breakdown of the errors is shown in Fig. 4 where the areas of the rectangles denote probabilities. If the two people are labeled A and B, then 20 errors are found by both A and B, 80 errors are found by A but not B, and 40 are found by B but not A.

If we consider only the 100 errors found by A, we see that 20 of them are found by B, which is a $1/5$ fraction. Since we are assuming that B finding a given error is independent of A finding it, we see that if B finds $1/5$ of the errors found by A, then he must find $1/5$ of the complete set of errors (on average). So $1/5$ is the probability that B finds any given error. Therefore, since we know that B found a total of 60 errors, the total number N of errors in the book must be given by $60/N = 1/5$. Thus $N = 300$. We can use similar reasoning applied to A to check this result. Namely $20/60 = 1/3$ is the probability that A finds any error, and so $100/N = 1/3$ also yields $N = 300$.

The unshaded region in Fig. 4 represents the errors that both people missed; it is $300 - 80 - 20 - 40 = 160$. ■.

5. **Relativistic Oscillator**

Conservation of energy is applicable in special relativity as long as we use the correct relativistic form of the energy of a particle. Say a relativistic particle of rest mass m_0 is acted upon by a restoring force $F_s = -kx$ and undergoes simple harmonic motion with total mechanical energy of $E_{\text{tot}} = \frac{1}{2}kx_0^2$. The position of the particle is denoted by $x(t)$.

(a) Using conservation of mechanical energy, write a first order differential for the particle's position (*Note:* Relativistic Kinetic Energy should be zero when the velocity is zero).

(b) Taking the amplitude of the oscillator to be x_0, write the period T of the oscillator as an integral. Write it in terms of $\omega = \sqrt{k/m_0}$.

(c) Take the non-relativistic limit (i.e., $kx_0^2/2 \ll m_0c^2$) and use the expansion $(1+x)^{-n} \simeq 1-nx+\mathcal{O}(x^2)$ to find the non-relativistic expression for this period.

Solution:

(a) By conservation of energy, we have $E_{\text{tot}} = T + V$ where V is the potential energy of a spring, $V = \frac{1}{2}kx^2$, and T is the relativistic kinetic energy of a particle of mass m_0:

$$T = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2. \tag{7}$$

We subtract away m_0c^2 to ensure that T has the proper non-relativistic limit $T = \frac{1}{2}mv^2 + \mathcal{O}(v^4/c^4)$. With this kinetic and potential energy and the original total energy, we find

$$\begin{aligned}
E_{\text{tot}} &= T + V \\
\frac{1}{2}kx_0^2 &= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 + \frac{1}{2}kx^2 \\
\frac{1}{2}k(x_0^2 - x^2) + m_0c^2 &= \frac{m_0c^2}{\sqrt{1-v^2/c^2}} \\
1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2) &= \frac{1}{\sqrt{1-v^2/c^2}} \\
\left[1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2)\right]^{-1} &= \sqrt{1-v^2/c^2},
\end{aligned} \tag{8}$$

which when we solve for $v = dx/dt$ yields

$$\frac{dx}{dt} = c\sqrt{1 - \left(1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2)\right)^{-2}}. \tag{9}$$

(b) To compute the period, we note that the mass with amplitude x_0 travels from $-x_0$ to x_0 in a time $T/2$ where T is the full period. Thus, separating the differential and integrating Eq.(9), gives us

$$\begin{aligned}
\frac{1}{c}\,dx\left[1 - \left(1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2)\right)^{-2}\right]^{-1/2} &= dt \\
\frac{1}{c}\int_{-x_0}^{x_0} dx\left[1 - \left(1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2)\right)^{-2}\right]^{-1/2} &= \int_0^{T/2} dt
\end{aligned} \tag{10}$$

and we find that the period is

$$T = \frac{2}{c}\int_{-x_0}^{x_0} dx \left[1 - \left(1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2)\right)^{-2}\right]^{-1/2}, \tag{11}$$

which (as far as I know) doesn't lead to any clean function.

(c) In the limit $kx_0^2/2 \ll m_0c^2$ or equivalently (with $k = m_0\omega^2$) $\omega^2 x_0^2 \ll c^2$ we can perform the expansion

$$\left(1 + \frac{\omega^2}{2c^2}(x_0^2 - x^2)\right)^{-2} = 1 - \frac{\omega^2}{c^2}(x_0^2 - x^2) + \cdots \tag{12}$$

where the dots stand for higher order terms. And so Eq.(11) becomes

$$\begin{aligned} T &= \frac{2}{c}\int_{-x_0}^{x_0} dx \left[1 - \left(1 - \frac{\omega^2}{c^2}(x_0^2 - x^2) + \cdots\right)\right]^{-1/2} \\ &= \frac{2}{c}\int_{-x_0}^{x_0} dx \left[\frac{\omega^2}{c^2}(x_0^2 - x^2)\right]^{-1/2} + \cdots \\ &= \frac{2}{\omega}\int_{-x_0}^{x_0} dx \frac{1}{\sqrt{x_0^2 - x^2}} + \cdots . \end{aligned} \tag{13}$$

After our expansion, the integral which defines the period is now soluble and we find

$$T = \frac{2}{\omega}\sin^{-1}\left(\frac{x}{x_0}\right)\Big|_{-x_0}^{x_0} + \cdots = \frac{2}{\omega}\left(\pi/2 - (-\pi/2)\right) + \cdots = \frac{2\pi}{\omega} + \cdots , \tag{14}$$

which is the expected non-relativistic result.

■

6. **Dependent Events**

   In Fig. 5, the ratio of an area of the square to the total square represents the probability for that particular space of events. From the figure, calculate the overall probability that B (but not necessarily *only* B) occurs.

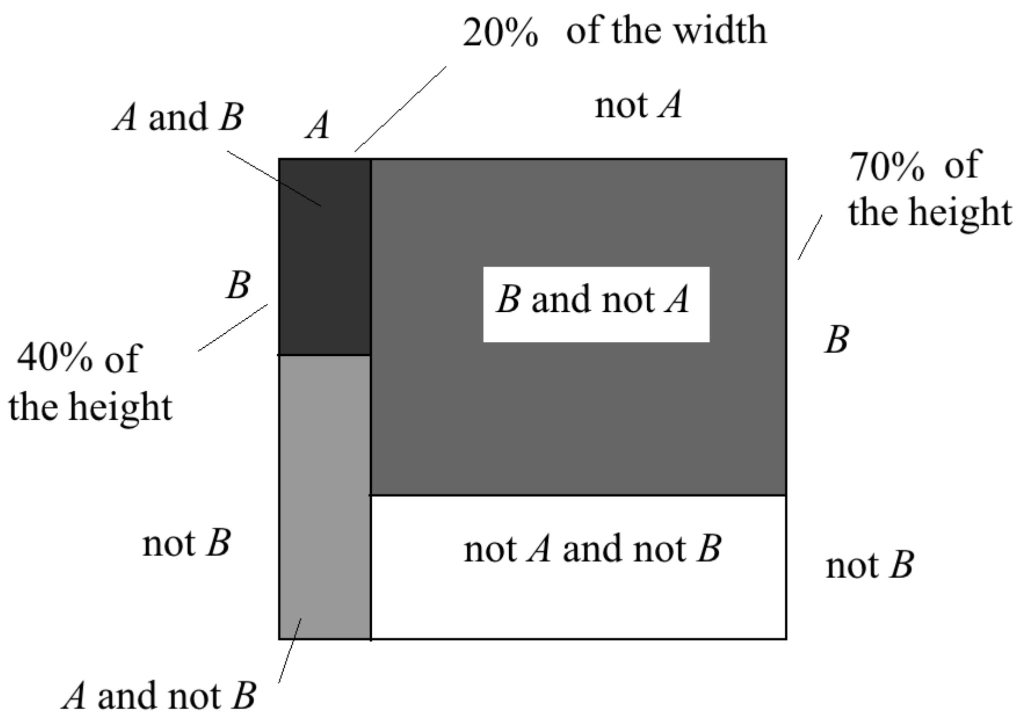

Figure 5: A hypothetical probability square

Solution: (transcribed from Morin's *Probability for the Enthusiastic Beginner*)

The problem is equivalent to finding the fraction of the total area that lies above the two horizontal line segments in Fig. 5. The upper left region is $40\% = 2/5$ of the area that lies to the left of the vertical line, which itself is $20\% = 1/5$ of the total area. And the upper right region is $70\% = 7/10$ of the area that lies to the right of the vertical line, which itself is $80\% = 4/5$ of the total area. The fraction of the total area that lies above the horizontal line segments is therefore.

$$\frac{1}{5} \cdot \frac{2}{5} + \frac{4}{5} \cdot \frac{7}{10} = \frac{2}{25} + \frac{14}{25} = \frac{16}{25} = 64\%. \tag{15}$$

■

7. **Waves on a Train**

The wave equation is a second-order partial differential equation describing oscillations which occur in both time and space. It has the form

$$\frac{\partial^2}{\partial t^2} f(x,t) = v_s^2 \frac{\partial^2}{\partial x^2} f(x,t), \tag{16}$$

where $f(x,t)$ is the physical quantity the wave describes and v is the speed of the wave. When the equation describes *traveling sinusoidal waves*, the corresponding solutions are of the form

$$f(x,t) = f_0 \cos(\omega t \pm kx) \quad \text{or} \quad f(x,t) = f_0 \sin(\omega t \pm kx), \tag{17}$$

where f_0 is the amplitude of the oscillations, ω is the angular frequency and k is the wave number. For such waves, the speed is defined as $v_s = \omega/k$.

A long bar is on a train moving with speed v. The bar lies with its axis parallel to the direction of motion of the train, and the entire bar undergoes simple harmonic motion with an amplitude h_0, such that its height can be described by the equation

$$h(x',t') = h_0 \cos(\omega t') \tag{18}$$

where x' is the horizontal position along the bar and t' is the time, both within the train frame.

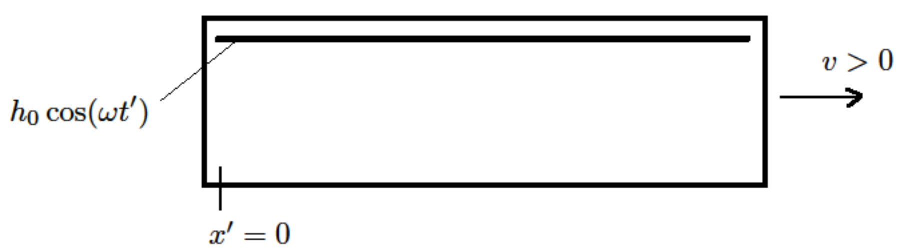

Figure 6: A long bar within a train undergoes simple harmonic motion *within* the train's frame.

Draw a picture and write an equation defining what this bar looks like from the ground frame. What is the speed of this phenomena from the ground frame? Why doesn't this speed violate the speed limit for massive objects?

Solution:

The main effect relevant here is that from the ground frame, clocks within a moving frame and which are farthest away from the front of that frame (as defined by the direction of its velocity) are a constant time *ahead* of clocks closer to the front of the moving frame. Thus in Fig. 6, from the ground frame we see the part of the bar at $x' = +\epsilon$ (for some small distance ϵ) as slightly behind in time the part of the bar at $x' = 0$. Similar results apply along the bar; a section of the bar at x is always seen at a time slightly behind the part at $x - \epsilon$ and slightly ahead of the part at $x + \epsilon$. Thus as we move to the right along the bar (again from the ground frame) the net effect of this lag, coupled with the oscillation of the bar itself, gives us a wave.

Being mathematically precise, we use the Lorentz Transformation

$$t' = \gamma(t - vx/c^2), \tag{19}$$

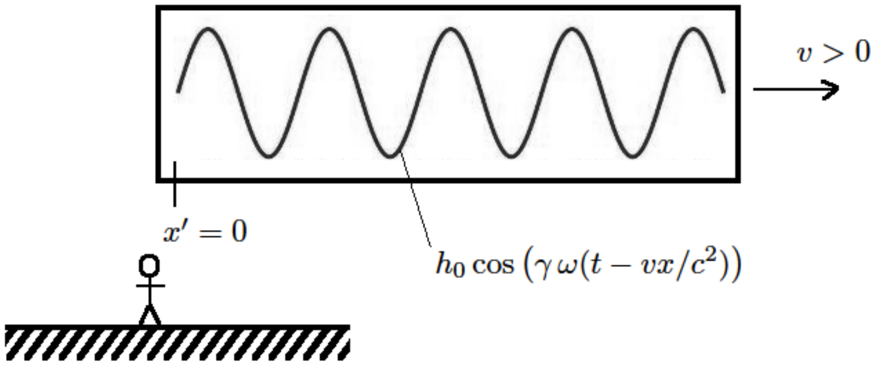

Figure 7

so that from the ground frame, the bar appears to act as a traveling wave defined by the equation

$$h(x', t') = h_0 \cos(\omega t') = h_0 \cos(\omega\gamma(t - xv/c^2)). \tag{20}$$

From the definition of Eq.(17), we know that the speed of this wave is

$$v_s = \frac{\omega\gamma}{\omega\gamma v/c^2} = \frac{c^2}{v} = \frac{c}{v} \cdot c > c, \tag{21}$$

which appears to suggest we have a physical object traveling faster than the speed of light. However, as Jacob mentioned in lecture, we can have two events which suggest superluminal travel as long as those two events are not causally linked (i.e., as long as one event doesn't cause the other). In this case, the wave propagation is not defined by any physical dynamics and does not consist of propagation through a medium. Hence although the wave travels at a speed faster than light, it does not violate the speed limit of special relativity because it does not define a causal phenomena.

■

8. **Birthday Problem**

Five random people are in a room together. Calculate the probability that at least 2 of them have the same birthday. Assume there are 365 days (no leap years) and each day has an equal probability of being any given person's birthday. (*Hint: It may be easier to calculate the probability that they all have different birthdays first.*)

Bonus: Can you find a general formula for the probability that at least two people have the same birthday in a room full of n random people?

Solution:

If P_{share} is the probability that at least two people share a birthday, then $P_{\text{share}} = 1 - P_{\text{don't share}}$ where $P_{\text{don't share}}$ is the probability that no two people share a birthday. We can say $P_{\text{don't share}} = P(1)P(2)P(3)P(4)P(5)$, where each $P(i)$ for $i = 1, 2, 3, 4, 5$ is the probability that the ith person does not share a birthday with the previously analyzed people.

Then we have $P(1) = 365/365$ because the first person can have any birthday. $P(2) = 364/365$ because person 2 can have any birthday except the birthday of person 1. $P(3) = 363/365$ because person 3 can have any birthday except the birthdays of person 1 and 2. And so on, and so forth. So we find

$$\begin{aligned} P_{\text{share}} &= 1 - P_{\text{don't share}} \\ &= 1 - P(1)P(2)P(3)P(4)P(5) \\ &= 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \\ &= 0.027. \end{aligned} \tag{22}$$

Thus there is a 2.7% probability that at least 2 people out of 5 have the same birthday.

Following the same procedure for the more general problem of n people, we obtain

$$\begin{aligned} P_{\text{share}} &= 1 - P_{\text{don't share}} \\ &= 1 - P(1)P(2)P(3)\cdots P(n) \\ &= 1 - \frac{365}{365} \cdots \frac{365-(n-1)}{365} \\ &= 1 - \left(\frac{1}{365}\right)^n 365 \cdot 364 \cdots \big(365-(n-1)\big). \end{aligned} \tag{23}$$

We can use this more general result to create a table of probability values as a function of n.

n	10	20	23	30	50	60	70	100
P_{share}	11.7%	41.1%	50.7%	70.6%	97%	99.4%	99.92%	99.99997%

Table 1

Thus, for our class of 34 people (staff and students), there is a greater than 70% chance that at least two people share the same birthday.

From the table we see the number where probability is half is 23, namely, when you have 23 people in a room, there is at least a 50% chance that at least two people have the same birthday.

The reason this result appears so surprising is explained by David Morin in *Probability for the Enthusiastic Beginner*

> One reason why many people can't believe the $n = 23$ result is that they're asking themselves a different question, namely, "How many people (in addition to me) need to be present in

order for there to be at least a $1/2$ chance that someone else has *my* birthday?" The answer to this question is indeed much larger than 23. The probability that no one out of n people has a birthday on a given day is simply $(364/365)^n$, because each person has a $365/365$ chance of not having that particular birthday. For $n = 252$, this is just over $1/2$. And for $n = 253$ it is just under $1/2$...Therefore, you need to come across 253 other people in order for the probability to be greater than $1/2$ that at least one of them *does* have your birthday (or any other *particular* birthday).

■

9. **Breast Cancer and Risk**

A woman comes in for a mammogram screening and is interested in her risk of having breast cancer. She is in a low-risk group where there is a 0.8% chance of having breast cancer. The doctor performs a test to determine if the woman has breast cancer. The test yields a result of positive 90% of the time when the tested woman does have breast cancer. It also yields a test of a positive (namely, a *false* positive) 7% of the time when the tested woman does not have breast cancer.

What is the probability that the woman (who is in a low-risk group) has breast cancer, given that her test results are positive?

Solution:

We will solve this problem intuitively and also more formally through Bayes' Theorem.

Intuitive Approach We take the stated probabilities to have been derived from frequentist averages according to our chosen sample population. We begin with a sample population of 1000 low-risk women. Of these 1000 women, $1000 \times 0.008 = 8$ women have breast cancer. Of these 8 women $8 \times 0.90 \sim 7$ would test positive for breast cancer under the screening.

Of the 992 women who do not have breast cancer, $992 \times 0.07 \sim 70$ women would get a false positive test for breast cancer. Thus the probability that a woman has breast cancer given that her test results are positive is

$$\begin{aligned} p \text{ of BC given (+) test} &= \frac{\text{\# of women who have BC and have a (+) test}}{\text{\# of women who have a (+) test}} \\ &= \frac{7}{7+70} \sim 0.09 \end{aligned} \tag{24}$$

So there is a 9% probability that the woman has breast cancer.

Bayesian We denote $p(B|+)$ as the probability to have breast cancer given a positive test; $p(+|B)$ the probability of getting a positive test given that one has breast cancer; $p(+)$ the probability of getting a positive test; $p(B)$ the probability of having breast cancer; $p(\neg B)$ the probability of not having breast cancer. Thus by the problem statement and Bayes' Theorem we obtain

$$\begin{aligned} p(B|+) &= p(+|B)\frac{p(B)}{p(+)} \\ &= p(+|B)\frac{p(B)}{p(+)} \\ &= p(+|B)\frac{p(B)}{p(+|B)p(B) + p(+|\neg B)p(\neg B)} \\ &= (0.90)\frac{0.008}{0.90 \times 0.008 + 0.07 \times 0.912} \\ &\simeq 0.09 \end{aligned} \tag{25}$$

■

10. **Monty-Hall Problem** You're on a game show where you want to win a prize car. You can win the car by guessing which door, out of three, the car is behind. Behind the other two doors are goats, which if chosen give you nothing. You choose door No.1, but before you look behind it, the game show host (who knows what is behind each door), reveals a goat behind one of the other doors. You still do not know what is behind door No.1, but the game show host now asks you if you would like to switch your chosen door to the one he did not reveal. Should you do it? (You want to maximize the probability that you find the car)

    Solution:

    Yes, you should switch. The probability that door No.1 contains the hidden car is 1/3, but the probability that door No.3 contains the hidden car (given that the host always reveals to you a goat) is 2/3.

    > *There is a 2/3 chance a goat is behind your chosen door No.1. The host always reveals to you the door containing the other goat. So if you switch to the hidden door, then (given that there's a 2/3 chance your initial choice was a goat), there is a 2/3 chance the hidden door contains the car.*

    Conceptual Argument: First, there is a probability of 1/3 that you chose the car when you selected door No.1. We can see this by noting that the game can begin with three different setups each of which is equally probable: we could have (C, G, G), (G, C, G), or (G, G, C) where the G stands for a goat, C for a car and the order of the component defines the number of the door the car or goat is behind. You always choose door No.1, and the car is behind door No.1 in only one out of three equally probable cases, so you have a one out of three (or 1/3) chance of choosing correctly.

    But when the game show host reveals to you the goat behind a certain door, then you now have more information about the situation. Now the possible situations look like

    - C: Your choice, G: Hidden Door, G: Revealed Goat
    - G: Your choice, C: Hidden Door, G: Revealed Goat
    - G: Your choice, G: Revealed Goat, C: Hidden Door

    In two our of three of these (again, equally probable) situations, the car is behind the unrevealed hidden door, so you have a 2/3 chance of choosing the car if you switch. Thus it is more advantageous to switch.

    > **Common Incorrect Solution:** An incorrect (but common solution) is to claim that the probability of choosing the car, after the host reveals the goat, is 1/2. This would only be correct if you came into the game not knowing what door you initially picked and only knowing the door that the host revealed. Rather, after the host reveals to you which of the doors you did not choose is the goat, it is now probabilistically more likely that the door you did not pick is a car.

    Bayesian Argument: You choose door No.1 from the start, and the host reveals a goat behind door No. 2. We want to know, given the hosts reveal, what is the probability that there is a car behind door No.3. (We note that this would be the same probability that there is a car behind door No.2, given the host reveals a goat behind door No.3) We denote the probability that there is a car behind No.3, given that the host reveals a goat behind No.2 as $p(\text{C, No.3}|\text{G, No.2})$. Then by Bayes Theorem

    $$\begin{aligned} p(\text{C, No.3}|\text{G, No.2}) &= \frac{p(\text{G, No.2}|\text{C, No.3})p(\text{C, No.3})}{p(\text{G, No.2})} \\ &= \frac{p(\text{G, No.2}|\text{C, No.3})p(\text{C, No.3})}{p(\text{G, No.2}|\text{C, No.3})p(\text{C, No.3}) + p(\text{G, No.2}|\text{C, not No.3})p(\text{C, not No.3})}. \end{aligned} \tag{26}$$

    We breakdown each probability in the above expression as follows

- $$p(\text{G, No.2}|\text{C, No.3}) = 1 \tag{27}$$

  If you choose No.1 as your first choice, and the car is behind No.3, then the host will *always* choose No.2 as the reveal. (He cannot reveal your own choice and he cannot reveal the car). Thus this probability is 1.

- $$p(\text{C, No.3}) = \frac{1}{3} \tag{28}$$

  There is a 1/3 chance the car is behind any particular door.

- $$p(\text{C, not No.3}) = \frac{2}{3} \tag{29}$$

  There is a 2/3 chance the car is *not* behind any particular door.

- $$\begin{aligned} p(\text{G, No.2}|\text{C, not No.3}) &= p(\text{G, No.2}|\text{C, No.1})p(\text{C, No.1}|\text{C, not No.3}) \\ &\quad + p(\text{G, No.2}|\text{C, No.2})p(\text{C, No.2}|\text{C, not No.3}) \\ &= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{4} \end{aligned} \tag{30}$$

  We want to know the probability that the host reveals a goat behind No.2 given that the car is *not* behind No.3. This probability is equivalent to the sum of the probability that the host reveals a goat behind No.2, given that a car is behind No.1 (which, again, itself is given that the car is not behind No.3) and the probability that the host reveals a goat behind No.2 given that a car is behind No.2 (which, again, itself is given that the car is not behind No.3).

  Given that the car is not behind No.3, there is a 1/2 probability that the car is behind No.1 (or No.2). So we have $p(\text{C, No.2}|\text{C, not No.3}) = p(\text{C, No.1}|\text{C, not No.3}) = 1/2$.

  If the car is behind No.2, then clearly the host cannot reveal a goat behind No.2.
  Thus $p(\text{G, No.2}|\text{C, No.2}) = 0$.

  But if the car is behind No.3, then the host has a 1/2 chance of revealing a goat behind No.2 and a 1/2 chance of revealing a goat behind No.3. Therefore $p(\text{G, No.2}|\text{C, No.1}) = 1/2$.

With these stated results, we find that Eq.(26) becomes

$$\begin{aligned} p(\text{C, No.3}|\text{G, No.2}) &= \frac{p(\text{G, No.2}|\text{C, No.3})p(\text{C, No.3})}{p(\text{G, No.2}|\text{C, No.3})p(\text{C, No.3}) + p(\text{G, No.2}|\text{C, not No.3})p(\text{C, not No.3})}. \\ &= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3}} \\ &= \frac{2}{3}, \end{aligned} \tag{31}$$

as we found using the more conceptual method above. Since there is a 2/3 chance the car is behind No.3 ($p(\text{C, No.3}|\text{G, No.2})$), given that the host reveal a goat behind No.2, and there was only a 1/3 chance chance that you chose the car behind No.1 ($p(\text{C, No.1})$), you should always switch to door No.3 (or whatever door the host did not reveal).

■