Section #5 Problems: Week of July 20

1. **Spinning and Falling** [1]

   A mass m is free to slide on a frictionless table and is connected, via a string of length ℓ that passes through a hole in the table, to a mass M that hangs below (see Fig. 1). Assume that M moves in a vertical line only, and assume that the string always remains taut.

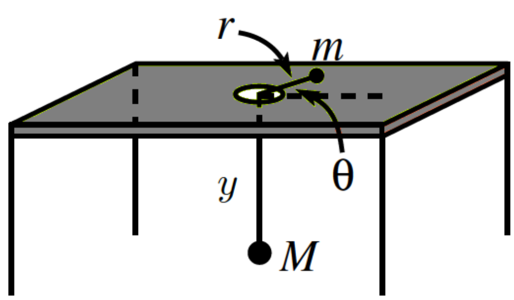

Figure 1: Figue from [1]. The variable r denotes the length of string on the table. The variable y denotes the length of string hanging through the hole. The total string has length ℓ.

   (a) Write down the Lagrangian in terms of the relevant kinematic variables in the figure. (Note that the mass on the table has both a *radial* and a*tangential* velocity).

   (b) (Consider this part only if you finish part (a)): What is the constraint for this system? Write it as $h = 0$ for a function h you must determine. .

   Solution:

   (a) The kinematic variables in the system are y which defines vertical motion, and r and θ which define motion within the plane of the table. We want to find the Lagrangian which is defined as

$$L = T - V = T_M + T_m - V, \tag{1}$$

   where T_M is the kinetic energy of mass M, T_m is the kinetic energy of mass m, and V is the potential energy of the system.

   The potential energy is the easiest to determine so we compute it first. Gravity is the only relevant external force in this system and it only directly acts on M, so the potential energy must be the gravitational potential energy of M. Given that y increases downward (according to the figure), and that gravitational potential energy must decrease the closer we get to earth's surface we have

$$V = -Mgy \tag{2}$$

   where we defined zero potential energy at the top of the table.

[1] Problem is from [1]

The kinetic energy comes in two contributions: the kinetic energy associated with mass M and the kinetic energy associated with mass m. The mass M moves only in a straight line so its kinetic energy is straight forward:

$$T_M = \frac{1}{2}Mv_y^2 = \frac{1}{2}M\dot{y}^2. \tag{3}$$

The mass m moves in a plane defined by the polar coordinates r and θ. The radial velocity is $v_r = \dot{r}$ and the tangential velocity is $v_\theta = r\dot{\theta}$. The kinetic energy of m is then

$$T_m = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2). \tag{4}$$

We could also reproduce this result by defining the vector $\vec{r} = (r\cos\theta, r\sin\theta)$, computing $\vec{v} = d\vec{r}/dt$, and then $T_m = \frac{1}{2}m\vec{v}^2$.

In all then, the Lagrangian of our system is

$$L = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + Mgy \tag{5}$$

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(b) In this system the total length of the string is fixed. Therefore, we have $y + r = \ell$ and writing this as $h = 0$ we have

$$h(r, y) = \ell - r - y = 0. \tag{6}$$

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2. **Walking on a Train-Part I**[2]

A train of length L (measured when it is at rest) and speed $v = 3c/5$ approaches a tunnel of length L. How long does it take (in the ground frame) for the train to pass completely through the tunnel assuming we start the clock when the front end enters the tunnel and we stop the clock when the back end exits the tunnel?

Solution:

We want to determine the time it takes a train to pass completely through a tunnel assuming we start the clock when the front end enters the tunnel and we stop the clock when the back end exits the tunnel. First the tunnel is stationary, so it's length as measured by us on the ground is simply L:

$$L_{\text{tunnel gr.}} = L \tag{7}$$

The train has a rest length L and it is moving at a velocity $3c/5$. Thus, from the ground frame, the length of the train is contracted to

$$L_{\text{train gr.}} = \frac{L}{\gamma}, \tag{8}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - (3/5)^2}} = \frac{1}{\sqrt{16/25}} = \frac{5}{4}. \tag{9}$$

The total distance L_{tot} covered by the back of the train during the time it takes the train to pass completely through the tunnel is a sum of the length of the tunnel and the length of the train, as measured from the ground frame. Thus

$$L_{\text{tot}} = L_{\text{train gr.}} + L_{\text{tunnel gr.}} = \frac{L}{\gamma} + L = L\left(1 + \frac{1}{\gamma}\right). \tag{10}$$

The time it takes the train to pass completely through the tunnel, is this length divided by the speed of the train. Thus we have

$$T = \frac{L}{v}\left(1 + \frac{1}{\gamma}\right) = \frac{5L}{3c}\left(1 + \frac{4}{5}\right) = \frac{3L}{c}. \tag{11}$$

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[2]Problem is from [1]

3. **Equation Diagramming**

   Each symbol used in an equation (particularly seemingly complicated ones) has meaning, and understanding the equation involves not only understanding how to apply it but also what each symbol means. We can develop this understanding by diagramming equations.

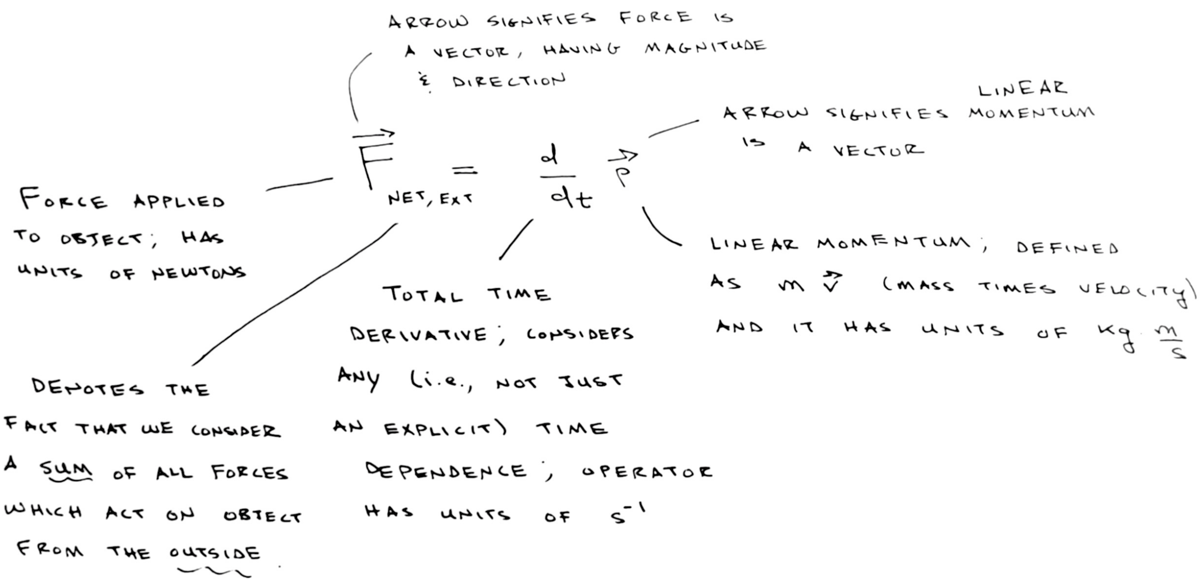

Figure 2: Equation Diagram for Newton's Second Law

   Using the equation diagram in Fig. 2 as an example, create analogous equation diagrams for the **Euler-Lagrange Equations** and **Hamilton's Equations**. Be as general as possible in the form of the equation (i.e., don't use a position variable when you should use a generalized coordinate variable) and be precise in how you write the equation and how you identify/describe the symbols in the equation.

   Solution:

   Solutions will vary. Possible ones are on the next page.

Lagrangian defined as *T-V* where *T* is kinetic and *V* is potential energy; has units of energy (Joules)

Acts like "acceleration" term

Partial derivative; only considers an explicit variable dependence

Total time derivative; considers *any* and not just an explicit time dependence; has units of s^(-1).

Result is only zero for systems *without* constraints

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$$

Acts like "force" term

Generalized coordinate velocity; units vary

Generalized coordinate; "alpha" accounts for multiple coordinates; units vary

Canonical Momentum; can define conserved quantities or be used to obtain the Hamiltonian

Figure 3: Equation Diagram for Euler-Lagrange Equations

Hamiltonian defines as *T+V* where *T* is kinetic and *V* is potential energy; has units of energy (Joules)

Partial derivative; only considers an explicit variable dependence

Total time derivative of canonical momentum; units vary

Acts like "p = mv"

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \qquad \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha$$

Acts like "F=ma"

Coordinate variable; units vary

Canonical momentum; units vary; "alpha" accounts for multiple coordinates

Total time derivative of coordinate variable; units vary

Time derivatives are total derivatives and thus act on any dependence in time

Figure 4: Equation Diagram for Hamilton's Equations

4. **From Here to There**[3]

A train of length L (measured when it is at rest) travels past you at speed v. A person on the train stands at the front, next to a clock that reads zero. At this moment in time (as measured by you), a clock ta the back of the train reads Lv/c^2. Evaluate whether the following statement can be true or not true. Explain either case you decide on.

> ”The person at the front of the train can leave the front right after the clock there reads zero, and then turn to the back and get there right before the clock there reads Lv/c^2. You (on the ground) will therefore see the person simultaneously at both the front and the back of the train when the clocks there read zero and Lv/c^2, respectively.”

Solution:

The solution to this problem rests on the fact that massive objects cannot travel faster than the speed of light and the "Rear Clock Ahead" effect.

Whenever a long object of length L is moving with a velocity v (in a direction parallel to its length) then the end which is farthest from the direction of motion (i.e., the *rear* end) is *ahead* in time by Lv/c^2 with respect to the time the end closest to the direction of motion (i.e., the front end). This was discussed in lecture and is called the "Rear Clock Ahead".

You on the ground see the back end of the train at a time Lv/c^2 into the future of the front end of the train. In order for you (on the ground) to see the person simultaneously at both the front and the back of the train, the person (in the train) needs to cover a distance L (the entire length of the train in his frame) in a time Lv/c^2. Thus the person needs to move a distance $\Delta x = L$ in a time $\Delta t = Lv/c^2$, which implies he must move with a speed

$$v = \frac{\Delta x}{\Delta t} = \frac{L}{Lv/c^2} = \frac{c}{v} \cdot c > 0. \tag{12}$$

The person cannot move at a speed faster than c, so this situation is not possible. ■

[3]Problem is from [1]

5. **What is the Question?**

Consider the physical situation depicted in the figure below. A rock is thrown at a speed v_0 and at an angle θ (from the horizontal) from the peak of a hill which slopes downward at an angle ϕ.

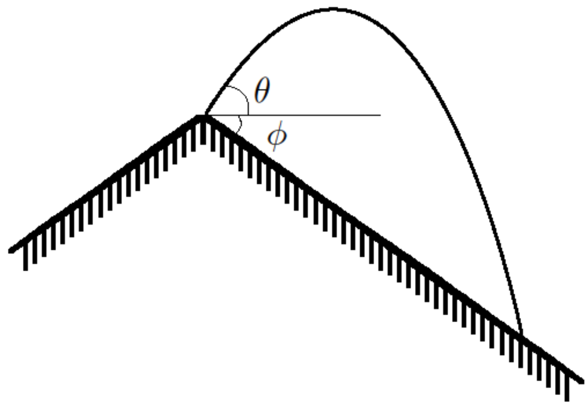

Figure 5: Projectile Motion

List **three physics-related** questions you can ask about this physical situation. Answer **two** of those questions, and write an outline of how you may answer the last question. Try to consider as many questions as possible.

Solution:

The purpose of this problem is to get you to practice asking questions about physical systems and formulating these questions in a way that they are answerable using the techniques you've learned. This is an open ended problem. Some of the questions one could ask are:

- How far along the decline does the rock travel?
- Assuming ϕ is fixed, what angle θ maximizes the total distance along the decline the rock travels?
- What is the velocity of the rock at the highest point in its trajectory?
- How far (vertically) below its starting point does the rock land?
- What is the potential energy of the rock at the lowest point in its trajectory?
- What is the total time it takes the rock to complete its trajectory?
- What angle maximizes the area under the curve of the trajectory?
- How would these results change if $\phi < 0$ (i.e., there was a decline instead of an incline).

We will answer the first two, namely we will determine the total distance along the decline the ball travels and determine the angle θ which maximizes this distance given a fixed ϕ.

For this problem the kinematical equations for x and y have their standard form

$$x(t) = v_0 \cos\theta t y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2. \tag{13}$$

If we take ℓ to be the total distance the object travels along the decline, then by geometry it has traveled a total horizontal distance $\ell\cos\phi$ and fallen a distance $\ell\sin\phi$ from its starting point. Assuming it completed its trajectory in a time t_f, we have the equations

$$x(t_f) = \ell\cos\phi = v_0\cos\theta t_f \tag{14}$$

$$y(t_f) = -\ell \sin\phi = v_0 \sin\theta t_f - \frac{1}{2} g t_f^2 \tag{15}$$

Solving Eq.(14) for t_f, we find

$$t_f = \frac{\ell \cos\phi}{v_0 \cos\theta}, \tag{16}$$

and plugging this result back into Eq.(15) gives us

$$-\ell \sin\phi = \ell \cos\phi \tan\theta - \frac{g\ell^2 \cos^2\phi}{2v_0^2 \cos^2\theta}. \tag{17}$$

Dividing by ℓ to eliminate the extraneous $\ell = 0$ solution we find

$$-\sin\phi = \cos\phi \tan\theta - \frac{g\ell \cos^2\phi}{2v_0^2 \cos^2\theta} \tag{18}$$

which when solved for ℓ gives

$$\ell = \frac{2v_0^2}{g} \frac{\cos^2\theta\,(\tan\phi + \tan\theta)}{\cos\phi}, \tag{19}$$

which is the total distance the rock travels along the decline. We note that, as we expect, $\phi = 0$ reduces to the standard $2v_0^2 \cos\theta \sin\theta / g$ result.

To compute the angle θ where this distance is maximum we differentiate Eq.(19) with respect to theta to find

$$\begin{aligned}
\frac{d\ell}{d\theta} &= \frac{2v_0^2}{g\cos\phi}\left[-2\cos\theta\sin\theta\,(\tan\phi + \tan\theta) + \cos^2\theta\left(\sec^2\theta\right)\right] \\
&= \frac{2v_0^2}{g\cos\phi}\left[-2\cos\theta\sin\theta\tan\phi - 2\sin^2\theta + 1\right] \\
&= \frac{2v_0^2}{g\cos\phi}\left[-\sin 2\theta \tan\phi + \cos 2\theta\right],
\end{aligned} \tag{20}$$

which implies Eq.(19) has a critical point at θ given by

$$\cot 2\theta_0 = \tan\phi. \tag{21}$$

Differentiating Eq.(20) once again and setting $\theta = \theta_0$, we find

$$\left.\frac{d^2\ell}{d\theta^2}\right|_{\theta=\theta_0} = -\frac{4v_0^2}{g\cos\phi}\left[\cos 2\theta_0 \tan\phi + \sin 2\theta_0\right] < 0. \tag{22}$$

Thus Eq.(21) defines the maximum length.

■

6. **Walking on a Train-Part II**[4]

A train of length L (measured when it is at rest) and speed $v = 3c/5$ approaches a tunnel of length L. At the moment the front of the train enters the tunnel, a person leaves the front of the train and walks (briskly) toward the back. She arrives at the back of the train right when it (the back) leaves the tunnel.

What is the person's speed with respect to the ground? (*Hint: You will need your answer from Walking on a Train-Part I*)

Solution:

In Part I of this problem, we found that it takes a time

$$T = \frac{L}{v}\left(1 + \frac{1}{\gamma}\right) = \frac{3L}{c} \tag{23}$$

for the train to pass completely through the tunnel of length L. If during this time a person leaves the front of the train *at the exact moment it enters the tunnel* and reaches the back of the train *at the exact moment it leaves the tunnel*, then the person has traveled a distance L as measured from the ground frame. (Try drawing a figure if this is unclear). The person has traveled a distance L in a time T so her speed is

$$v_0 = \frac{L}{T} = v\left(1 + \frac{1}{\gamma}\right)^{-1} = \frac{c}{3}. \tag{24}$$

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[4]Problem is from [1]

7. **Concept Mapping**

A concept map is a diagram which shows how distinct ideas are related to one another. As an example consider the concept map below.

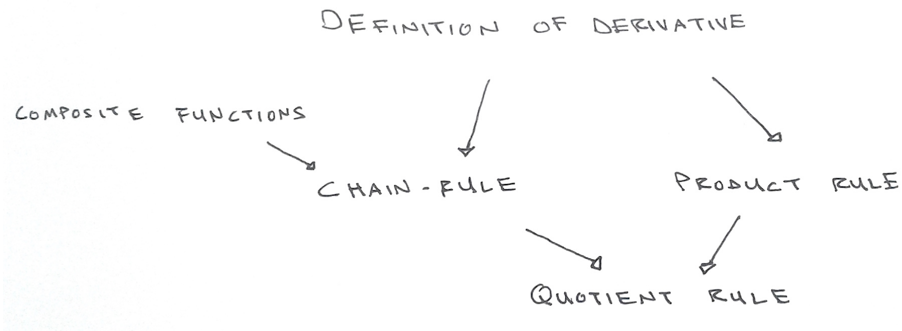

Figure 6: Example of equation diagram.

It indicates the following logical sequence of derivations: The quotient rule of differentiation can be derived from a combination of the chain-rule and the product-rule; the chain-rule arises from the definition of composite functions and the definition of the derivative; and the product rule comes from the definition of the derivative. (*The arrows denote the fact that whatever is at the "tail" of the arrow is used to derive whatever is at the "head" of the arrow.)

Using the example in Fig. 6 as an example, create a concept map (with arrows) showing how the definition of kinetic and potential energy leads to conservation of momentum through the Lagrangian formalism. Use the following terms as discrete nodes in the concept map:

- Kinetic Energy
- Potential Energy
- Noether's Theorem
- Definition of Lagrangian
- Definition ofAction
- Definition of Dynamical Symmetry
- Conservation Laws
- Dynamical Symmetry under Angular Translations
- Conservation of Angular Momentum

If you have time associate an equation with each of these terms within the concept map.

Solution:

Solutions can vary and can be variously argued. Here is a possible map:

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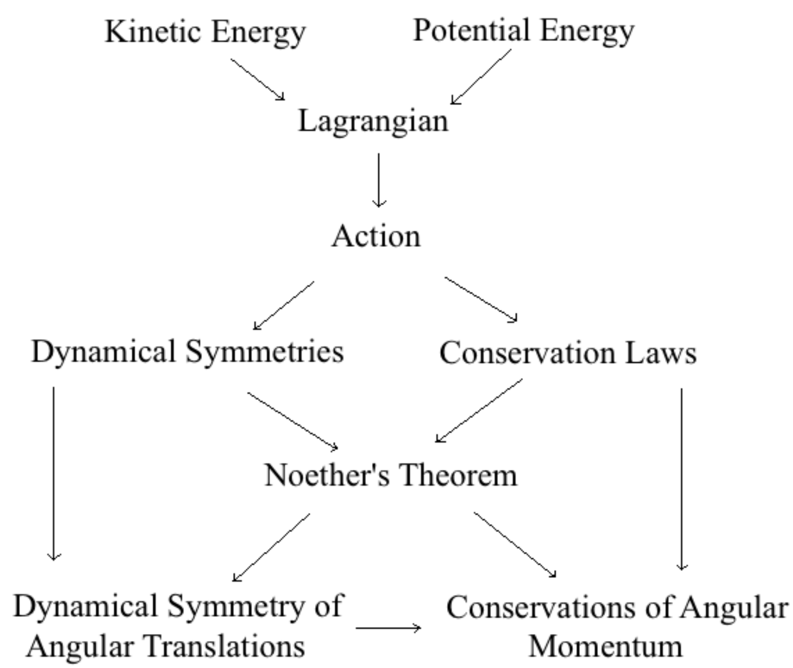

Figure 7: Conservation of Energy Concept Map

8. **Projectile Motion on a Train**

A boy is on a train that is moving with velocity V. Within the train, he throws a ball with speed v_0 at an angle θ from zero height and sees that it completes a full trajectory. From the ground frame, what is the horizontal distance the ball travels? Again from the ground frame, how long does it take the ball to complete it's trajectory? (*Hint: You should use a Lorentz Transformation for this problem.*)

Solution: We recall that for a frame S' moving at a velocity V relative to a frame S, we can transform the spatial distance $\Delta x'$ and duration $\Delta t'$ in S' to the corresponding quantities in S through the transformations

$$\Delta x = \gamma(\Delta x' + V\Delta t') \tag{25}$$
$$\Delta t = \gamma(\Delta t' + V\Delta x'/c^2), \tag{26}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$. From the basic kinematic equations for projectile motion:

$$x(t) = v_0 \cos\theta t \tag{27}$$
$$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2, \tag{28}$$

we know we can derive that the total time $\Delta t'$ it takes to complete the trajectory and the total horizontal distance $\Delta x'$ of the trajectory is

$$\Delta t' = \frac{2v_0}{g}\sin\theta \tag{29}$$
$$\Delta x' = \frac{2v_0^2}{g}\cos\theta\sin\theta. \tag{30}$$

Therefore, by Eq.(25), the total horizontal distance covered by the projectile according to a person on the ground is

$$\Delta x = \gamma\frac{2v_0}{g}\sin\theta(v_0\cos\theta + V). \tag{31}$$

And the total time it takes to complete the trajectory, again as measured by someone on the ground is

$$\Delta t = \gamma\frac{2v_0}{g}\sin\theta(1 + Vv_0\cos\theta/c^2). \tag{32}$$

■

Side Note:

We note that $\Delta x/\Delta t$ gives us what we expect (by velocity addition) for the horizontal velocity of the projectile as seen from the ground.

$$\frac{\Delta x}{\Delta t} = \frac{v_0\cos\theta + V}{1 + Vv_0\cos\theta/c^2}. \tag{33}$$

9. **Atwood and Hamilton** [5]

Consider the Atwood's machine, in the figure below. Let x be the vertical position of the left mass, with upward taken to be positive. Find the Hamiltonian in terms of x and its conjugate momentum, and write down Hamilton's equations.

(If you have time write down the solution assuming $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$)

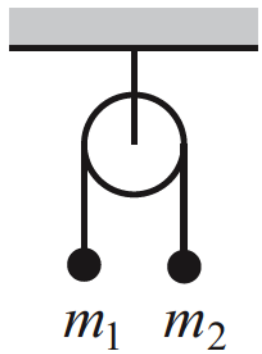

Figure 8: Atwood Machine from online chapter of [1]

Solution: For this problem we will use the definition of the Hamiltonian H in terms of the Lagrangian L

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L, \tag{34}$$

where

$$p_\alpha = \partial L / \partial \dot{q}_\alpha \tag{35}$$

is the canonical momentum, to find the Hamiltonian of the system, and then apply Hamilton's equations

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \tag{36}$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{37}$$

to find the equations of motion.

First, the Lagrangian. If we let x denote the height from the ground of m_1 then if we increase x the height of m_2 decreases. Only changes in potential energy are physically relevant so we can take the potential energy of m_1 to be $V_1 = +m_1 g x$ and we can take the potential energy of m_2 to be $V_2 = -m_2 g x$.

Also, the two masses are connected by a fixed string of length ℓ, and thus they move together and the speed of each mass is the same $|\dot{x}|$. Therefore, the kinetic energy of the first mass is $T_1 = \frac{1}{2} m_1 \dot{x}^2$ and the kinetic energy of the second mass is $T_2 = \frac{1}{2} m_2 \dot{x}^2$. The Lagrangian of this system is then

$$L = T_1 + T_2 - V_1 - V_2 = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 - m_1 g x + m_2 g x = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2) g x. \tag{38}$$

[5] Problem is from online chapter of [1]

To find the Hamiltonian we first compute Eq.(35) to find the momentum of this system in terms of the velocity. We have

$$p = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x}. \tag{39}$$

The Hamiltonian is only a function of momenta and positions, so for this system we need to express velocities in terms of momenta. Therefore by Eq.(34) the Hamiltonian is

$$\begin{aligned} H &= p\dot{x} - L \\ &= p\frac{p}{m_1 + m_2} - \frac{1}{2}(m_1 + m_2)\left(\frac{p}{m_1 + m_2}\right)^2 + (m_1 - m_2)gx \\ &= \frac{p^2}{m_1 + m_2} - \frac{p^2}{2(m_1 + m_2)} + (m_1 - m_2)gx \\ &= \frac{p^2}{2(m_1 + m_2)} + (m_1 - m_2)gx \end{aligned} \tag{40}$$

Next the Equations of Motion. Applying Hamilton's equations to this Hamiltonian gives us

$$\frac{\partial H}{\partial p} = \dot{x} \quad \Longrightarrow \quad \frac{p}{m_1 + m_2} = \dot{x} \tag{41}$$

$$\frac{\partial H}{\partial x} = -\dot{p} \quad \Longrightarrow \quad (m_1 - m_2)g = -\dot{p}. \tag{42}$$

Which is the desired answer. Eliminating $\dot{p}$ from this system gives us the x equation of motion

$$\ddot{x} = -\frac{(m_1 - m_2)g}{m_1 + m_2} \tag{43}$$

If we were to solve these equations for the kinematics of our system we would find

$$x(t) = x_0 + \dot{x}_0 t - \frac{(m_1 - m_2)}{2(m_1 + m_2)}gt^2. \tag{44}$$

This result makes conceptual sense. The acceleration of $x(t)$ is only positive if $m_2 > m_1$, and if $m_2 = m_1$ we expect there to be no acceleration because of the balancing gravitational forces.

■

10. **Practicing Lorentz Transformations**

In a certain inertial frame S, two events are observed to occur at the same place, but occur L/c apart in time. In a different inertial frame S' the same two events are observed to occur $L/2$ apart in space.

What is the time between the two events in S'? (*Hint: You should use a Lorentz Transformation.*)

Solution:

For a frame S' moving at a velocity v relative to a frame S, we can transform the spatial distance $\Delta x'$ and duration $\Delta t'$ in S' to the corresponding quantities in S through the transformation

$$\Delta x = \gamma(\Delta x' + v\Delta t') \tag{45}$$
$$\Delta t = \gamma(\Delta t' + v\Delta x'/c^2), \tag{46}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Our goal in this problem is to find $\Delta t'$ by first finding v.

For the system in this problem, $\Delta t = L/c$ and $\Delta x = 0$. Thus from Eq.(45) and Eq.(25) we have

$$0 = \gamma(\Delta x' + v\Delta t') \tag{47}$$
$$\frac{L}{c} = \gamma(\Delta t' + v\Delta x'/c^2). \tag{48}$$

From the first equality we have

$$\Delta t' = -\frac{1}{v}\Delta x' \tag{49}$$

which when plugged into the second equality gives us

$$\begin{aligned}\frac{L}{c} &= \gamma\left(-\frac{1}{v} + \frac{v}{c^2}\right)\Delta x' \\ &= -\gamma\left(1 - \frac{v^2}{c^2}\right)\frac{\Delta x'}{v} \\ &= -\gamma\frac{1}{\gamma^2}\frac{\Delta x'}{v} = -\frac{1}{\gamma}\frac{\Delta x'}{v}.\end{aligned} \tag{50}$$

But from the problem statement we know $\Delta x' = L/2$, so our equality becomes

$$\frac{L}{c} = -\frac{L}{2\gamma v} \tag{51}$$

or, equivalently,

$$2 = \frac{\sqrt{1 - v^2/c^2}}{v/c}. \tag{52}$$

Squaring both sides of this equation, we have

$$4 = \frac{1 - v^2/c^2}{v^2/c^2} = \frac{c^2}{v^2} - 1, \tag{53}$$

and solving for v gives us

$$v = -\frac{c}{\sqrt{5}}, \tag{54}$$

where we take the negative solution because Eq.(49) must be positive. Returning to Eq.(49) and using $\Delta x' = L/2$, we find

$$\Delta t' = -\frac{1}{v}\Delta x' = \frac{L}{2c\sqrt{5}} \tag{55}$$

References

[1] D. Morin, *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.