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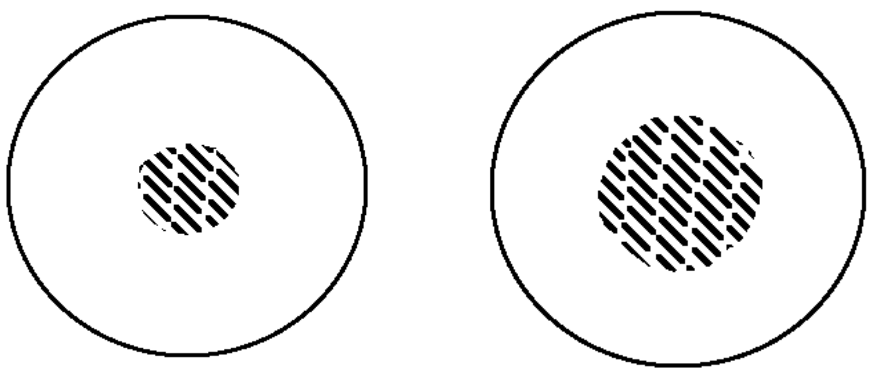
Structure of Knowledge in Physics

The theoretical results in physics are taken as valid not specifically because they are communicated and accepted by authority figures, but because—in addition to being confirmed by our observations, as is true in all scientific fields—they are logically and, specifically, mathematically consistent. It is the mathematical consistency of these theories which largely defines what I call the "structure of physics" and it is this structure which allows the student to be able to develop and understand much of theoretical underpinnings of the subject without recourse to any established authority[1]. In these notes we discuss this structure and why it implies physics should be approached and learned with more deeper techniques than just memorization.

1 Model Building in the Lab

Imagine you're a biology student who has acquired a solid understanding of calculus but is woefully underread in the history and understood science of biology. You make up (or, at least, try to make up) for these deficiencies by being quite inventive and logical.

You're working in a bacteriology lab for the summer. Your formal task is to answer some question about the metabolic properties of a bacteria (which we call 'bacteria α'), but you instead become intrigued by how the bacteria are growing. Their colonies ensconced in nutrient plates seem to be expanding before your eyes.



(a) Bacterial colony at time t_0

(b) Bacterial colony at time $t_0 + 1$ hr

Figure 1: Bacterial growth in nutrient plates. The aggregate of pill shaped lines represents the bacteria colony and the large circular border represents the outline of the nutrient plates in which the bacteria grow.

During the first hour, you **observe** the bacteria and measure the diameter of the roughly circular shape the colony makes on the nutrient plate. You take these measurements at the start and at the end of the hour, and you notice that the ratio between the final diameter $d(t_0 + 1\text{ hr})$ and the initial diameter $d(t_0)$ is 1.411. You do this again in the next hour and you find that the ratio of the diameters at the end and the beginning of the hour is again 1.411. Given this (admittedly small) sample size, you conclude that the value of this ratio is constant in time. Having mastered geometry and algebra, you recognize this ratio is, within an error, $\sqrt{2}$, and hence you conclude $d(t_0 + 1\text{ hr})/d(t_0) = \sqrt{2}$ independent of t_0. You also recognize that if the diameters

[1] Assuming, of course, that said student had the necessary technical background, was able to ask the right questions, and knew what assumptions to make, none of which is easy to obtain/acquire

are related by a factor of $\sqrt{2}$, then the areas A at each point in time are related by a factor of 2. Encapsulating your knowledge mathematically, you conclude

$$\frac{A(t_0 + 1\text{ hr})}{A(t_0)} = 2. \tag{1}$$

You also know that whenever it takes a quantity a fixed amount of time to double in number, the growth rate of the quantity must be proportional to the quantity itself (Think of compound interest for example). Thus you decide to elevate your observations to the level of a "principle for bacterial growth". You postulate

> **Principle of Bacteria Growth:** If a colony of bacteria α is in a culture dish with sufficient nutrients, the growth rate of the colony's area is proportional to the area itself.

Initially satisfied with this formulation, you put down your pen and start heading out to an early lunch. But your calculus knowledge pulls you back. Your newly formulated "principle of bacteria growth" is expressed qualitatively, but its language suggests a quantitative formulation. Taking $A(t)$ to be the area of the bacteria colony at a time t and $dA(t)/dt$ to be the growth rate of that area at that same time t, you realize your principle of bacteria growth can be written mathematically as

> **Principle of Bacteria Growth (Mathematical Formulation):** If a colony of bacteria α is in a culture dish with sufficient nutrients, then the area A of the colony evolves in time according to
>
> $$\frac{dA(t)}{dt} = kA(t), \tag{2}$$
>
> for some k of dimensions $1/\text{time}$.

Eq.(2) seems like an improvement over the previous principle, however it contains an ambiguity in the introduction of k. Of course the original principle also contained this ambiguity, but because of its qualitative formulation the ambiguity was less apparent. Fortunately, you surmise that it should be possible to determine the value of k by making proper use of your previous. Namely, given Eq.(1), you want to know what the theory represented by Eq.(2) predicts for how long it takes the area of the bacterial colony to double; this prediction will in turn constrain the value of k.

Again, given your calculus knowledge, you recognize that the general solution to Eq.(2), is

$$A(t) = A_0 e^{kt}, \tag{3}$$

where A_0 is the area of the colony at the chosen initial time $t = 0$. By Eq.(3), after a doubling time of t_d, the area of the colony must be $2A_0$. Thus you find

$$2 = \frac{A(t_d)}{A_0} = \frac{A_0 e^{kt_d}}{A_0} = e^{kt_d}. \tag{4}$$

Taking the far LHS and the far RHS and solving k gives you $k = \ln 2/t_d$. With your previous observation that $t_d = 1$ hr, you then find

$$k = \ln 2\text{ hr}^{-1}. \tag{5}$$

With this result, Eq.(2) is mathematically precise and specific to our system of study.

You now want to see what else you can do with this principle. Knowing that the bacteria are in a finite circular plate of diameter 10 cm, you realize there is a constraint in this setup on how much the bacteria can grow according to Eq.(2). You decide to test this by predicting how long it would take the bacteria (which currently comprises an area with diameter of 3 cm) to reach the limits of the culture dish. Given Eq.(3) and

Eq.(5), you **predict** it should take a time

$$t_f = \frac{1}{k} \ln \frac{A_f}{A_0} \approx 3.5 \text{ hr}, \tag{6}$$

where $A_f/A_0 = (10/3)^2$, to reach the limits of the dish. You look at your watch to check the time and finally decide to leave the lab and go on a very long lunch. When you return...

Well you (reader) get the idea. This example was mean to illustrate the dual processes of induction and deduction and how they allow us to mathematically model and make predictions about the world. Moreover, although this example doesn't at all deal with physics, through its foundation in mathematical modeling it provides a conceptually simple framework by which to develop a structural understanding of physics.

2 Structure of Physics

- **Induction and Deduction:** Induction refers to the process of developing or postulating general laws based on specific observations. Deduction refers to the process of deriving specific predictions from general laws.

  Induction and deduction are both often seen as fundamental to the progress of science, and although there are arguments for why the induction/deduction dual does not characterize all of science, it does largely describe how physicists formulate theories. In the previous section, a theory of bacterial growth was formulated by first observing the growth of bacteria, extrapolating a principle from the properties of this growth (induction), and then using the principle to make a testable prediction (deduction). We depict the cyclical nature of this process below.

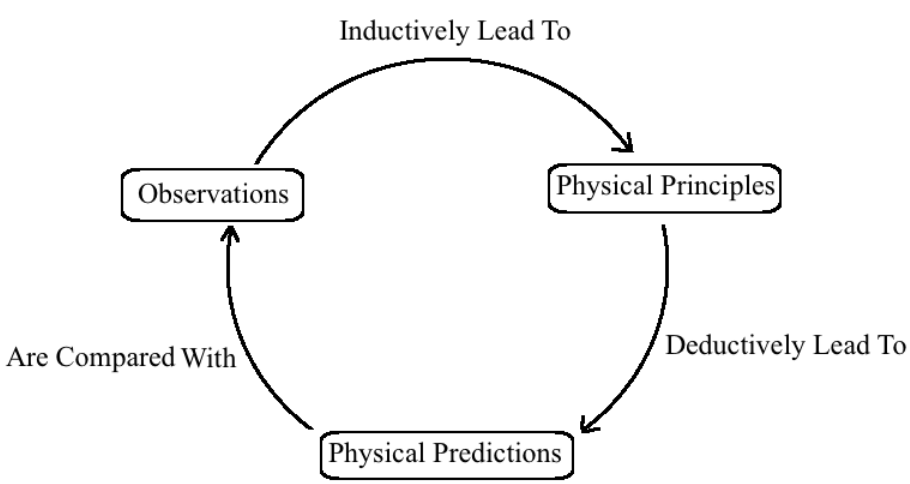


Figure 2: The cyclical relationship between the observations which motivate the development of principles which are in turn used to obtain predictions which are checked against more observations.

  It is important to take note of this structure because it largely characterizes how physicists in all disciplines formulate theories. For example, Newton's laws are principles (gleaned from observations of physical phenomena) which produce predictions which can be compared with other observations. We should mention that this process in practice is rarely ever this clean, and the physicist often jumps between stages as he works through trial-and-error to find the proper principles or predictions by which to model a phenomena.

Also, sometimes physicists don't directly use observations to develop physical principles but rely on abstracted intuition. Einstein's formulations of Special and General Relativity began with such an intuition [1].

Finally, the principles of one theory can exist as the predictions of another, and so the division between what is a principle and what is a prediction is not always clear. For example, Kepler's laws can be seen either as the principles of pre-Newtonian astronomy or the predictions of Newtonian astronomy, and Newton's law of gravitation could be seen as a principle of Newtonian gravitation or as a prediction of Einstein's theory of General Relativity.

- **Principles and Predictions:** This example was also meant to illustrate the difference (with regard to a hierarchy of importance) between principles and predictions. Principles are taken as assumptions and are often used as the starting points of a theory. From these starting points one extends the theory in various directions to obtain predictions (see Fig. 3 below). In this way we consider principles as more fundamental than predictions, but understanding a theory often amounts to understanding both the principles and predictions in addition to the ways they are connected.

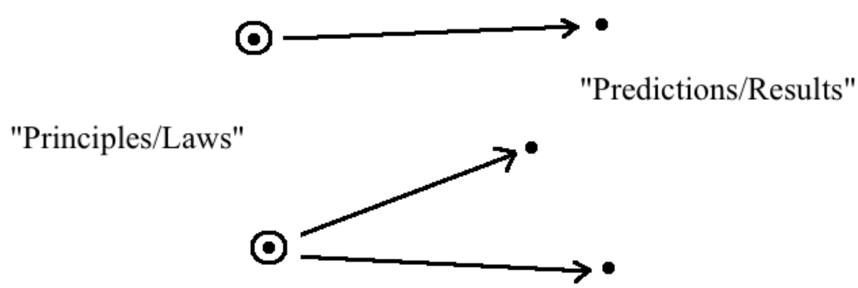


Figure 3: Predictions in a theory extend from and are less "fundamental" than the principles. This does not mean predictions are less important; only that they are not the deductive starting point of the theory. In this diagram the arrows stand for a mathematical derivation.

We can illustrate the relationship between predictions and principles with a non-physics related example. Consider the following valid logical argument:

1. A is true.
2. If A is true, then B is true.
3. B is true

In this argument, statement 1 serves the role of a premise which means it is not derived from any other statement; to consider the validity of this argument, we simply take the premise as true. Similarly, in physics, physical principles and laws are taken as true without any valid logical justification[2].

Statement 3 is the conclusion of this logical argument since it is deduced from the premise. The conclusion of a logical argument is analogous to a physical prediction or derived result in physics because

[2] Again, there are sometimes exceptions to this if we look at the relationship between some subjects in physics

these physical predictions are deduced from physical principles.

What role does statement 2 play? In the argument, it acts as a logical connection between statements 1 and 3 and thus requires us to accept the truth of statement 3 if we accept the truth of statement 1. In physics, mathematics–and more specifically the logic implied by the mathematical framework of a physical theory–serves the role of statement 2 as it is used to derived predictions from principles.

- **The Role of Mathematics:** Some people state that mathematics is the language of physics to emphasize that, in a way very similar to how written language can be used to express qualitative ideas, mathematics can be used to express ideas about change, geometry, and structure—ideas which are often relevant to the study of physical systems. However, in physics, mathematics is more than just a symbolic method for expressing physical ideas; it also serves to embed these idea in a sophisticated logical framework which in turn allows these ideas to be connected to other ideas.

  We see this most clearly in the way physical principles expressed mathematically allow us to derive predictions for a physical theory. In Fig. 3, the arrows represent a mathematical derivation in the physical theory. In the (non-physics) example in the first section, by using calculus to mathematically express our "principle of bacterial growth" we were able to predict how long it should take the bacteria to overflow the container. More generally, the mathematics we use to make predictions in a theory vary by subject. To illustrate this relationship, in the table below we list a few physical theories along with some of their associated physical predictions and the mathematical frameworks used to derived them.

Table 1: Physical Theories: Principles and the Mathematics used to obtain Predictions

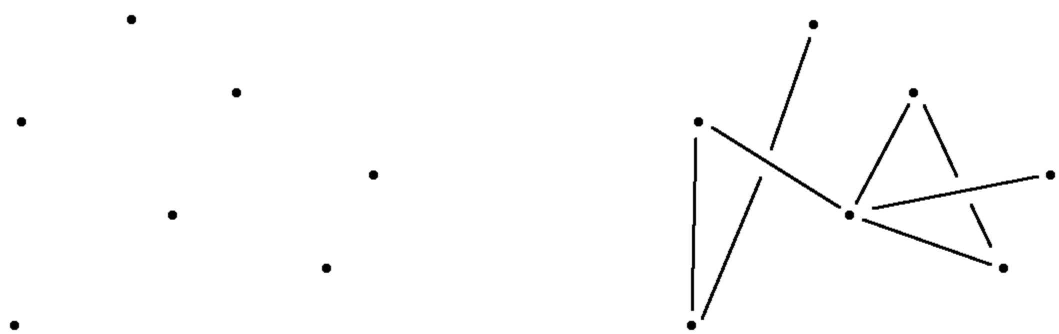
	Principles	Mathematics	Predictions/Results
Classical Mechanics	Newton's Laws; Principle of Least Action; Newton's Law of Gravitation	Calculus; Functional Calculus; Differential Equations	Period of Pendulum; Conic Section Orbits
Electromagnetism	Maxwell's Equations; Lorentz Force Law	Vector Calculus; Partial Differential Equations	Electric field of General Conductor; Energy Radiated by Electric Charge
Quantum Mechanics	Schrödinger Equation; Definition of State Kets and Hilbert Space; Interpretation	Calculus; Partial Differential Equations; Linear Algebra	Energy Levels of Hydrogen Atom; Van der Waals forces

  I want to emphasize that the principles listed in this table are not absolute principles in all areas of physics, and instead there are some subjects for which these principles exist as predictions. Indeed much of the work considered to be foundational in physics, (for example, the work done here at Harvard in The Center for Fundamental Laws of Nature) consists of deriving current principles from even more fundamental ones.

3 Considering and Learning Physics

This discussion of the structure of physics is important because it motivates a particular way of studying and engaging with the subject. In particular, it should discourage an unfortunately typical way of learning the subject and encourage a way which is more natural and in the long run more useful.

In some physics classes, the main results which define a physics subject are sometimes employed and practiced in a disconnected manner which suggests that these results were found and hence should be applied independently of one another. A standard example is "The Big Four (or Five)"➡ equations of kinematics which are presented as distinct. However, these kinematic equation can all naturally be subsumed into the equation for constant acceleration. This is general in physics; all results in a physical theory (outside the physical principles and assumptions) are derived from and hence connected to other results. We depict this perspective on physics in Fig. 4



(a) Taking ideas and equations as distinct and learning them as such. Leads to memorization and an inability to extend knowledge.

(b) Understanding connections between ideas and equations. Allows for less memorization and develops skills to extend knowledge.

Figure 4: Two types of understanding

Fig. 4 is meant to suggest that what you learn in a subject becomes more useful when you understand how topics and ideas are connected. One benefit of this way of learning is that if you forget something in a subject (e.g., imagine if a node in Fig. 4 was erased), then since you understand how results in the subject are connected, you can rederive what you have forgotten from all that you still know.

To put it plainly: only memorizing equations is an inefficient way to learn a subject which has as much manifest logical structure as physics does. Instead, efficiently learning physics entails getting a sense of the structure of the subject and of how theorems and physical results are derived and are related to one another.

Therefore understanding in physics is not merely represented by what equations or ideas you know, but more truly in what you know about the connections between these equations and ideas. It is only by understanding these existing connections that you will ever have the knowledge and skills to move beyond them toward a comprehension of something you have never before seen.

References

[1] A. Einstein, "On the method of theoretical physics," *Philosophy of science*, vol. 1, no. 2, pp. 163–169, 1934.