Checking Your Work:

Units, Limiting Cases, and Mathematics as Metropolis

As part of any physics class you spend a lot of time answering questions, deriving physical results, and predicting the consequences of experiments both real and imagined. In short you do a lot of analytical work and thereby obtain equations which supposedly describe reality. But how can you be sure? Well, to be precise, it is impossible (even with apparent experimental confirmation) to be absolutely sure that our physical theories correspond to something truly fundamental about our world, but we can often perform simple logical checks without a laboratory to ensure that the associated results are not obviously wrong.

In these notes we discuss two useful checks in this direction in addition to a more general heuristic for checking one's work. The specific definitions of these checks are given here, but you won't be missing anything if you first look at the example in the next section.

Units/Dimensions Check: Since all physical quantities have units and dimensions[1] any expression which represents a physical quantity must have the same units/dimensions as that quantity. This check uses this fact to ensure that an analytic expression has the correct physical units/dimensions. Taking "$[A]$" to stand for the units[2] of A we have,

$$\text{If } f(a_1, \ldots, a_N) \text{ represents the physical quantity } F, \text{ then } [F] = [f(a_1, \ldots, a_N)] \tag{1}$$

Limiting Case Check: This check rests on the idea that the mathematical result modeling a physical phenomena should have analytical properties which mirror the qualitative properties of the phenomena. Expressed mathematically, if we have an analytic function f (with parameters $a_1, \ldots, a_N$ which represents a physical quantity F, then we must have

$$\lim_{a_i \to A} f(a_1, \ldots, a_N) \;=\; \text{What we expect for } F \text{ when } a_i = A. \tag{2}$$

If either of these checks fail (i.e., the expected limiting cases are not satisfied or $[F] \neq [f]$), then we know our result is incorrect.

1 Example Application

Both of these techniques are best illustrated through examples. We'll consider one example here but there are many more in chapter 1 of [1] which can be found in the Physics Library on the fourth floor of Jefferson. We begin with the following problem statement:

> A time dependent force $F(t) = F_0 e^{-bt}$ acts in the x direction on a mass m. The particle has an initial position x_0 and an initial velocity of $\dot{x}_0$. The particle's position $x(t)$ is
>
> $$x(t) = x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m}(e^{-bt} - 1). \tag{3}$$

[1] Dimensions are either Mass, Length, Time (or Charge). Units refer to quantities like Newton's, Joules, etc.

[2] In physics, $[A]$ typically stands for the *dimension* of A, but we'll only be considering units here so we appropriate this notation.

1. **Units Check:** Given that F_0 has the units of force, b has the units of s^{-1}, and m has the units of kg, check that $x(t)$ has the correct units.

2. **Limiting Cases Check:** Consider $x(t)$ in the limits of:

   (a) $F_0 \to 0$

   (b) $m \to \infty$

   (c) $b \to \infty$

   Do these limits of $x(t)$ yield the expected analytic expressions?

1.1 Units Check

Recall that **units** refer to what is typically known as SI units (quantities like Newtons, Joule's, Watts, etc.) while **dimensions** refer to the fundamental quantities length, mass, and time. Units can be expressed in terms of dimensions so we can check either the units or the dimensions of an equation; both must be consistent across the equality for the equation to make sense.

We will consider units here because that is what we typically use to express the physical nature of quantities. We are trying to check the units of the equation

$$x(t) = x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m}(e^{-bt} - 1). \tag{4}$$

The equation represents the position of a particle, and position has units of meters so for the left hand side of the equation we have

$$\Big[x(t)\Big] = \mathrm{m}. \tag{5}$$

This unit expression must be matched on the right hand side of the equation. Namely, we must find

$$\Big[x_0 + \dot{x}_0 t + \frac{F_0}{bm} t + \frac{F_0}{b^2 m}(e^{-bt} - 1)\Big] = \mathrm{m} \tag{6}$$

Transcendental functions like e^x have no units, so demonstrating this unit equivalence amounts to showing that the quantities

$$x_0\,, \quad \dot{x}_0 t\,, \quad \frac{F_0}{bm} t\,, \quad \frac{F_0}{b^2 m} \tag{7}$$

have units of meters.

The first quantity represents initial position, so the units are automatically manifest:

$$\Big[x_0\Big] = \mathrm{m}. \tag{8}$$

The next quantity is a velocity multiplied by a time, so we have

$$\Big[\dot{x}_0 t\Big] = \Big[\dot{x}_0\Big] \times \Big[t\Big] = \frac{\mathrm{m}}{\mathrm{s}} \times \mathrm{s} = \mathrm{m}, \tag{9}$$

as we expect. By Newtons' Second Law, force must have the same units as mass times acceleration. Thus the units of F_0 are kg·m/s^2. Also, because the argument of the exponential e^{-bt} must not have any units (i.e., transcendental functions must be functions of real numbers alone), the units of b are s^{-1}. The units of the final two terms in Eq.(3) are then

$$\Big[F_0 t/bm\Big] = \Big[F_0\Big] \times \Big[t\Big] \times \Big[\frac{1}{b}\Big] \times \Big[\frac{1}{m}\Big] = \mathrm{kg}\,\frac{\mathrm{m}}{\mathrm{s}^2} \times \mathrm{s} \times \frac{1}{\mathrm{s}^{-1}} \times \frac{1}{\mathrm{kg}} = \mathrm{m} \tag{10}$$

$$\left[F_0/b^2m\right] = \left[F_0\right] \times \left[\frac{1}{b^2}\right] \times \left[\frac{1}{m}\right] = \text{kg}\,\frac{\text{m}}{\text{s}^2} \times \frac{1}{\text{s}^{-2}} \times \frac{1}{\text{kg}} = \text{m} \tag{11}$$

In all we find Eq.(3) has the correct units.

1.2 Limiting Cases

To consider the limiting cases we apply Eq.(2) to investigate the various limits listed in the problem statement.

1. $F_0 \to 0$: If we take $F_0 \to 0$ then the force applied to the particle goes to zero, and we expect the particle to move with constant velocity in time. Taking this limit for Eq.(3) this expectation is confirmed.

$$\lim_{F_0\to 0} x(t) = \lim_{F_0\to 0}\left[x_0 + \dot{x}_0 t + \frac{F_0}{bm}t + \frac{F_0}{b^2m}(e^{-bt}-1)\right] = x_0 + \dot{x}_0 t \tag{12}$$

2. $m \to \infty$: If we take the particle's mass m to infinity, then we expect a finite force $F(t)$ to not change the particle's velocity. Essentially, the particle would appear to not be acted upon a force at all, and we would again expect the particle to move with constant velocity in time. We indeed find this is the case:

$$\lim_{m\to \infty} x(t) = \lim_{F_0\to 0}\left[x_0 + \dot{x}_0 t + \frac{F_0}{bm}t + \frac{F_0}{b^2m}(e^{-bt}-1)\right] = x_0 + \dot{x}_0 t \tag{13}$$

3. $b \to 0$: In the function $F(t) = F_0 e^{-bt}$ the parameter b acts as a time constant for the decaying-in-time force. Specifically $\tau = 1/b$ is the amount of time it takes the force to decay to $1/e$ of its initial value. If $b \to 0$ then this decay time τ goes to infinity and the force never decays in time. In other words, it is simply a constant force. As a constant force, it gives the particle a constant acceleration and we expect the position as a function of time to have the form

$$x(t) \stackrel{?}{=} x_0 + \dot{x}_0 t + \frac{F_0}{2m}t^2. \tag{14}$$

To check this result we focus on the final two terms in Eq.(3). Taking the limit and making use of the Taylor expansion of the exponential we find

$$\begin{aligned}\lim_{b\to 0} \frac{F_0}{bm}t + \frac{F_0}{b^2m}(e^{-bt}-1) &= \lim_{b\to 0} \frac{F_0}{bm}t + \frac{F_0}{b^2m}\left(-bt + \tfrac{1}{2}b^2t^2 + \mathcal{O}(b^3)\right)\\ &= \lim_{b\to 0} \frac{F_0}{bm}t - \frac{F_0}{bm}t + \frac{F_0}{m}t\tfrac{1}{2}t^2 + \mathcal{O}(b)\\ &= \frac{F_0}{2m}t^2,\end{aligned} \tag{15}$$

and so Eq.(14) is established. We note we could have obtained this result without an explicit Taylor expansion by applying L'hopital's rule twice.

As a logical point, although Eq.(3) is indeed correct, whenever we apply a limiting case or dimensions check and find that both checks are consistent, at most what we can say is that our checks prove the result is *not definitely wrong* (instead of proving that the result is *definitely correct*). Namely, it is possible for an expression to pass limiting cases and dimensions check and not be the correct result for a physical system.

The Takeaway: When you obtain a final equation for a problem, check that both the units of the problem and limiting cases are what you expect. Given the generality of these two checks, they can be used in any context which involves mathematical modeling of some physical phenomena.

2 Mathematics as Metropolis

All fields of mathematics are like well-connected cities in which there is often more than one way to get from one point to another. Physics, which employs mathematics as an integral language, greatly benefits from this connectivity. You all saw an example of this when the Lagrangian and Hamiltonian formulations of mechanics were presented this week. The three approaches to classical mechanics you've seen so far (i.e., the Newtonian, Lagrangian, and Hamiltonian approahces) reveal that distinct, but related, physical principles can be used to derive the dynamics of a system, and each approach uniquely informs our understanding of the system.

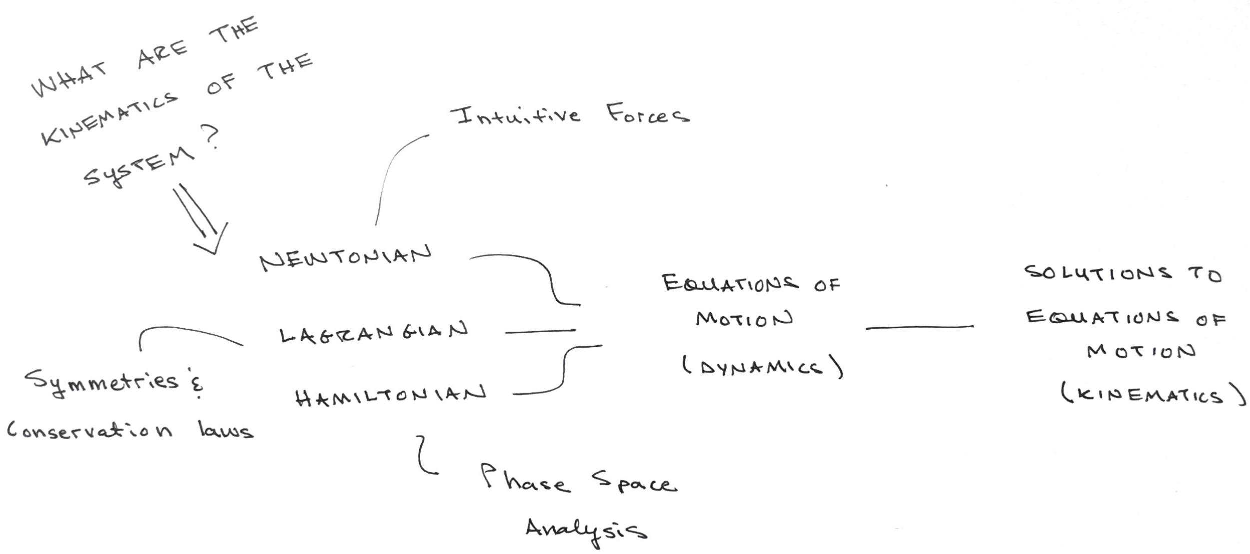

Figure 1: Various Approaches to Mechanics: The three approaches to mechanics we've learned so far all allow us to obtain the dynamical equations of a system (which in turn leads to the same kinematics), but each give us unique ways to characterize and describe that dynamics.

Less macroscopically, even within a single physical formulation, it's possible to make use of physics's intrinsic connectivity by searching for multiple ways to solve a problem. This is useful because when you have two different ways of approaching the same problem, it's like you have two different people working on the problem, and if both approaches are logically and physically correct they should each result in the same answer.

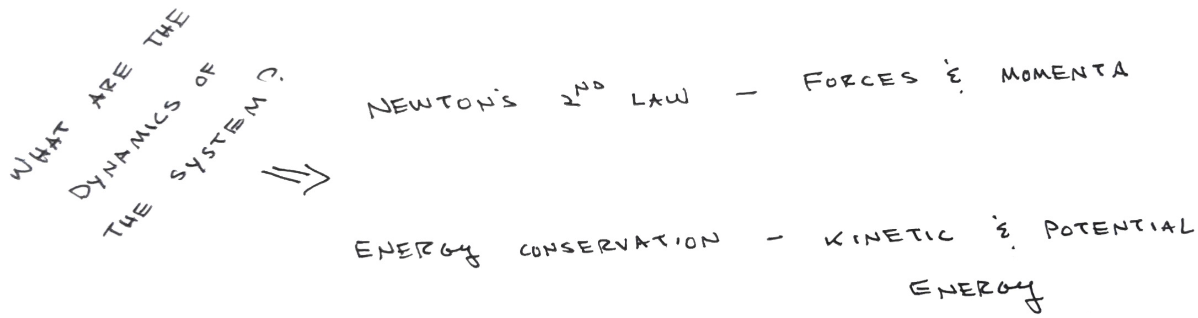

Figure 2: Within Newtonian Mechanics we can define the dynamics of a system according to Newton's Second Law or energy conservation, the latter of which can be seen as a consequence of the former.

2.1 Multiple Approaches to a Problem: Example

As an example of considering multiple approaches to a problem, we'll look at something which should be familiar: vertical motion in a gravitational field. For such a system, energy is conserved and thus it is possible to describe the dynamics by energy conservation as well as by Newton's Second Law.

Specifically, we will show that both of these dynamical perspectives lead to the same kinematical equations for the vertical motion of a mass m in a gravitational field.

2.1.1 First Approach: Newton's Second Law

Using Newton's Second Law, we begin with the dynamical equation

$$m\ddot{z}(t) = F_{\text{ext}, z} = -mg \tag{16}$$

where $z(t)$ defines the vertical position of the particle, m is the mass, and g is the gravitational acceleration. To find the kinematical equation $z(t)$ describing the particle's motion, the *standard* procedure is to divide by m and integrate this equation twice in time. For the first integration, we have

$$\begin{aligned} \int_0^t dt'\, \ddot{z}(t') &= \int_0^t dt'\, \frac{F_{\text{ext}, z}}{m} \\ \dot{z}(t')\Big|_0^t &= -\int_0^t dt'\, g \\ \dot{z}(t) - \dot{z}_0 &= -gt. \end{aligned} \tag{17}$$

And for the second integration, we have

$$\begin{aligned} \int_0^t dt'\, \dot{z}(t') &= \int_0^t dt'\, (\dot{z}_0 - gt') \\ z(t')\Big|_0^t &= \left(\dot{z}_0 t - \frac{1}{2}gt'^2\right)\Big|_0^t \\ z(t) - z_0 &= \dot{z}_0 t - \frac{1}{2}gt^2. \end{aligned} \tag{18}$$

Which is the desired result.

2.1.2 Second Approach: Energy Conservation

We could have derived this result using conservation of energy as well. For such a system, conservation of energy requires that the total mechanical energy E_0 remains the same throughout the mass's trajectory:

$$E_0 = \frac{1}{2}m\dot{z}(t)^2 + mgz(t) = \frac{1}{2}m\dot{z}_0^2 + mgz_0 \tag{19}$$

where $\dot{z}_0$ and z_0 are the initial velocity and position respectively. Solving for $\dot{z}$ in the first equality in Eq.(19), we find

$$\dot{z}(t) = \frac{dz}{dt} = \sqrt{2(E_0/m - gz)}. \tag{20}$$

This equation can be rearranged and integrated to obtain

$$\int_0^t dt' = \int_{z_0}^{z(t)} dz' \frac{1}{\sqrt{2(E_0/m - gz')}}. \tag{21}$$

The integral on the left hand side is just t. The integral on the right hand side has a straight forward result which can of course be checked by differentiation:

$$\int_{z_0}^{z(t)} dz' \frac{1}{\sqrt{2(E_0/m - gz')}} = -\frac{1}{g}\sqrt{2(E_0/m - gz')}\Big|_{z_0}^{z(t)}. \tag{22}$$

We therefore find for Eq.(21),

$$\begin{aligned} -gt &= \sqrt{2(E_0/m - gz(t))} - \sqrt{2(E_0/m - gz_0)} \\ -gt &= \sqrt{2(E_0/m - gz(t))} - \sqrt{2(\dot{z}_0^2/2)} \\ \dot{z}_0 - gt &= \sqrt{2(E_0/m - gz(t))}. \end{aligned} \tag{23}$$

Squaring both sides of this expression and using $E_0/m = gz_0 + \frac{1}{2}\dot{z}_0^2$, we find

$$\dot{z}_0^2 - 2g\dot{z}_0 gt + g^2t^2 = 2gz_0 + \dot{z}_0^2 - 2gz(t). \tag{24}$$

Finally, solving for $z(t)$ gives us

$$\begin{aligned} z(t) &= -\frac{1}{2g}\Big(- 2gz_0 - \dot{z}_0^2 + \dot{z}_0^2 - 2g\dot{z}gt + g^2t^2\Big) \\ &= z_0 + \dot{z}_0 t - \frac{1}{2}gt^2 \end{aligned} \tag{25}$$

as expected.

The Takeaway: When you solve a problem using one physical or mathematical approach, try to think of other approaches both to check your answer and to obtain a different perspective on the problem.

References

[1] D. Morin, *Introduction to classical mechanics: with problems and solutions*. Cambridge University Press, 2008.