Derivations as Arguments

In these notes we explain why derivations should not be memorized step-for-step, but should be understood according to their underlying ideas and the mathematical framework used to render these ideas. The specific calculations a derivation employs are often context dependent and will change according to who is performing the calculation (or even when they're performing it). But if the central idea which runs through the calculation is correct and well understood, it can be be applied consistently. And so when reading through a long derivation, it is better to try to understand the foundational ideas of the derivation (i.e., being able to answer all the "Why?"s) instead of trying to memorize its specific steps.

1 Symbols versus Ideas

1.1 An Argument Analogy

We will start with a passage from Neil Postman's *Amusing Ourselves to Death* [1].

> What Orwell feared were those who would ban books. What Huxley feared was that there would be no reason to ban a book, for there would be no one who wanted to read one. Orwell feared those who would deprive us of information. Huxley feared those who would give us so much that we would be reduced to passivity and egoism. Orwell feared that the truth would be concealed from us. Huxley feared the truth would be drowned in a sea of irrelevance. Orwell feared we would become a captive culture. Huxley feared we would become a trivial culture. . . . In *1984*, Huxley added, people are controlled by inflicting pain. In *Brave New World*, they are controlled by inflicting pleasure. In short, Orwell feared that what we hate will ruin us. Huxley feared that what we love will ruin us.

Let us say you're a student who wants to understand this passage. Where exactly is the meaning of this passage contained, and how can you show that you understand this meaning in a way that allows you to generalize it and apply it to new contexts?

One way to approach finding meaning is to rewrite word-for-word the entire passage and to memorize each sentence in the exact order. But such an approach is a rather shallow way of reconstructing meaning and is in fact much more difficult than the more sensible approach of understanding the idea of the argument and then reproducing the idea using personally familiar language (i.e., in the oft-quoted "you own words").

Moreover reconstructing an argument word-for-word is a particularly transparent indication that you really don't understand it; doing so indicates that you think the paragraph's meaning is tied to the exact words it uses instead of the ideas it conveys. And without a deeper understanding of the ideas, you're essentially doomed to be incapable of applying the argument to new contexts.

In short, the meaning in the passage lies in its ideas for which the words are merely a proxy for communication, and understanding the passage is predicated on understanding the ideas and how they are connected to and used to argue the conclusion.

1.2 Derivations in Physics

This point has a clear analog in studying derivations in physics. In the same way that copying and regurgitating an argument word-for-word suggests a lack of understanding of that argument, copying and regurgitating a derivation symbol-by-symbol suggests a lack of understanding of that derivation.

Let's take the derivation that a launch angle of 45° maximizes the total horizontal distance traveled in two-dimensional projectile motion:

Maximum Horizontal Distance for Two-Dimensional Projectile Motion

To compute the angle which maximizes the total horizontal distance traveled by a projectile we will calculate the total horizontal distance the projectile travels as a function of the launch angle, and then apply the maximization algorithm from calculus to find the optimizing angle.

We begin with the parametric equations for two-dimensional projectile motion:

$$y(t) = v_0 \sin\theta t - \frac{1}{2}gt^2 \tag{1}$$

$$x(t) = v_0 \cos\theta t, \tag{2}$$

where v_0 is the initial velocity, θ_0 is the launch angle, g is the gravitational acceleration and t is the time. In Eq.(1) and Eq.(2) we assumed the particle began at $t = 0$ at $(x, y) = (0, 0)$, at zero horizontal and vertical position respectively. We define the time at the end of the projectile's trajectory to be t_f. At this time, the particle should be at the height $y = 0$, and so we have

$$y(t_f) = v_0 \sin\theta t_f - \frac{1}{2}gt_f^2 = 0. \tag{3}$$

Solving for the non-zero t_f gives us $t_f = 2v_0/g \sin\theta$. Inserting this result into Eq.(2), gives us the total horizontal distance D as a function of θ:

$$x(t_f) = \frac{2v_0^2}{g}\sin\theta\cos\theta = \frac{v_0^2}{g}\sin 2\theta \equiv D(\theta). \tag{4}$$

To maximize this function we find the angle θ_1 for which $D'(\theta_1) = 0$ and for which $D''(\theta_1) > 0$. Doing so we have the calculations

$$D'(\theta = \theta_1) = \frac{2v_0^2}{g}\cos 2\theta_1 = 0 \quad \longrightarrow \quad 2\theta_1 = \pi/2, 3\pi/2 \tag{5}$$

$$D''(\theta = \theta_1) = -\frac{4v_0^2}{g}\sin 2\theta_1 < 0 \quad \longrightarrow \quad 2\theta_1 = \pi/2, \tag{6}$$

which imply θ_1, the angle which maximizes the total horizontal travel distance, is $\pi/4$ or $45°$.

Now, to study this derivation a student can try to memorize each step and the exact order in which the steps are written. But, again, such an approach would be a reflection of a lack of understanding as it would indicate that the meaning lies only in the specific steps the derivation uses.

Rather, in order to understand the derivation, our proverbial student should try to understand not merely its specific steps, but *why* these steps were taken, namely the ideas that underlie them. Understanding the underlying ideas of derivation often requires being able to answer three types of questions: What, How, and Why.

To Deconstruct a Physics Derivation Ask:

- **What:** What question is the derivation trying to answer? What are the starting assumptions or starting points for the derivation?
- **How:** How is this question answered? Specifically, what mathematical framework and techniques do you need for the derivation?

- **Why:** Why is the question the derivation answers even being asked?[a]?

[a] Understanding why the question is being asked gives you a more global justification for the derivation which in turn allows you to understand how the question fits into the larger objectives of the subject. This gives you a sense of what similar questions you can ask which in turn extends your understanding of the derivation by relating it to possible others.

In this case the answer to these questions are

- **What:** The derivation seeks to determine the angle which maximizes the horizontal distance traveled by a projectile. The starting points are the kinematical equations for horizontal and vertical motion in a gravitational field.
- **How:** The question is answered by applying algebra, trigonometry, and the maximization algorithm in calculus.
- **Why:** This derivation answers an important question in kinematics; It allows us to know what angle we need to launch any projectile (assuming low air-resistance) in order to cover the greatest horizontal distance. It also allows us to setup (at least) the analogous procedure for finding the launch angles which maximize travel time, enclosed area, and trajectory arc-length.

With answers to these questions, one also better understands the purpose behind the distinct steps of the derivation and could therefore better construct the derivation himself/herself. Also, knowing the underlying ideas of a derivation allows one to extrapolate the methods of the derivation to new contexts.

1.3 What you get when you keep asking ”Why?”

On the first day of class, you all were asked what defines physics. Someone[1] answered that physics was ”What you get when you keep asking ’Why?’”. This is an excellent definition because it illustrates that continually asking why leads to a deeper and more fundamental understanding of our physical world. But the benefits of such questioning are universal and are also found in asking ”Why?” about physical derivations.

If there is one tactic that summarizes the ethos of this section it is that one should ask ”Why” about everything.

The Takeaway: When you are working through a mathematical argument or generally any problem, ask ”Why?” about *everything* (i.e, every step, series of steps, assumptions, result). Having an answer to all the ”Why?”s gives you a more complete understanding of the problem.

2 Relationship to Knowledge

The point of this discussion is to encourage the student to develop a more sophisticated relationship to what we typically define as “truth”. The relationship between student and truth as it is codified in knowledge is one which lies in the background of all education but is rarely ever discussed. Throughout their education, students obtain most of their knowledge of the world from textbooks, teachers, and other standards of authority Fig. 1.

There is a reason for this somewhat veiled method of indoctrination. There is a large body of scientific work accumulated over decades and centuries and in order for new scholars to successfully extend and use this knowledge, it is simply more efficient for them to take this knowledge as true without working through the confusions and missteps that previous practitioners had to go through in order to build it.

However, eventually students will come to a point in their education or work where they are interested in answering a question which has not been discussed by the authority figures or has been discussed in a

[1] Daniel Glazer

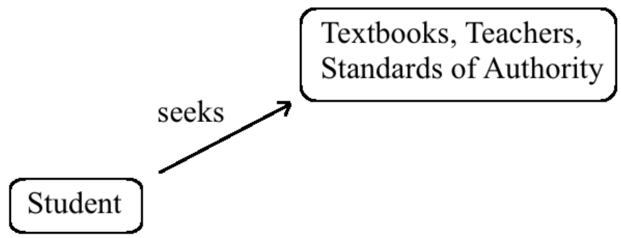

Figure 1: Students' conception of truth early on in their education. In such a conception, students largely act as a "consumer of knowledge".

way that is unsatisfactory. In such a situation, the student is forced into a more sophisticated relationship with knowledge[2] in which teachers, books, and other standards of authority touch upon the sought for knowledge but do not completely encompass it.

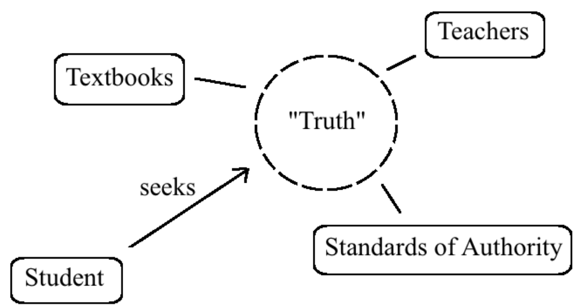

Figure 2: Students' conception of truth when they are forced to move beyond their initial education. In such a conception the student largely acts as a "producer of knowledge".

The student can use these previously canonized standards of authority to obtain particular perspectives or interpretations on the motivating question, but ultimately what is sought is something which lies beyond these external representations.

> **No Stupid Questions:** The student's more sophisticated relationship with truth is one reason why it is not possible for a student to sincerely ask a "stupid question". The apparent objectivity of science often obscures the fact that it is done by human beings who are motivated by mostly subjective reasons. The networks of knowledge people construct—networks which depend on what they have focused on learning, what they think is important and interesting, etc.—largely reflects what they value. Consequently, any question which extends from such personal values is a legitimate one even if there seems to be no objective justification for it. In fact, historically, it is clear that the development of science is largely predicated on the the scientific community's tolerance and support for a scientist's personally motivated questions, many of which others might call naive or even "stupid".[2].

Because of your anticipated future relationship with knowledge, we urge you to practice developing a

[2] We label this knowledge as "truth" in the figure, but there are philosophical reasons why it is not possible for humans to precisely discern the objective reality typically definitive of truth; we ignore these reasons.

deeper understanding of derivations and arguments so that when you eventually need to develop your own, you have already built the reasoning skills which allow you to not be limited by the current understanding of a question.

References

[1] N. Postman, *Amusing ourselves to death: Public discourse in the age of show business*. Penguin, 2006.

[2] R. Root-Bernstein, *Discovering. Inventing and solving problems at the frontiers of science*. Cambridge, MA: Harvard University Press, 1989.