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Continuum limit of mean-field partition functions

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Abstract

In physics, the continuum limit consists of transforming discretely-spaced degrees of freedom on a lattice into smoothly-varying fields on a continuous domain. In these transformations, the degrees of freedom are typically dynamical variables like position or probability. However, in statistical physics, partition functions themselves can be seen as existing on a discrete lattice whose individual sites are identified by the number of degrees of freedom in the system. In this work, we pursue this interpretation and show that the continuum limit of a certain class of mean-field theory partition functions yields a partial differential equation whose solution provides large N approximations to the partition function of the original system. The equation is obtained by promoting the number of degrees of freedom to a continuous variable and then re-interpreting (as finite differences in a continuous space) the recursive relation that defines the partition function. For some example systems, we show that solutions to this equation yield transition temperatures with the same parameter-scaling behavior as those of the original system. We conclude by discussing how this formalism can motivate 'diffusion in degree-of-freedom space' interpretations of how interacting partition functions vary with temperature.

Keywords: continuum systems, partition functions, partial differential equations, phase transitions, stirling numbers

1. Introduction

In physics, the 'continuum limit' is a way to simplify the dynamical equations of a system by promoting a discrete index to a continuous variable, introducing a lattice spacing between the new continuous variables, and, finally, taking the lattice spacing to zero. By the end of these transformations, finite differences are converted to derivatives, the discrete system has become a continuous one, and the many dynamical equations for the original variables have been reduced to a simpler set of dynamical equations for a new variable with an argument of one-higher dimensionality.

The canonical application of the continuum limit is to a chain of harmonic oscillators. For masses with positions $x_j(t)$ that are connected end-to-end by springs with frequency ω_0, we have the dynamical equation

$$\ddot{\mathbf{x}} = -\hat{\Omega}\mathbf{x}, \tag{1}$$

where $\mathbf{x}^T = (x_1, x_2, \ldots, x_N)$, and the elements of the matrix $\hat{\Omega}$ are $\hat{\Omega}_{i,j} = \omega_0^2(2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1})$ [Pai05]. Taking masses j and $j+1$ to be separated by a lattice spacing and then taking that spacing to zero converts the system of springs to that of a continuous string, and the associated dynamical equations change from a system of N second-order equations to a single partial differential equation. Analytically, the transformation converts a discrete eigenvalue problem to a Fourier series problem. Physically, the transformation reveals how expanding the scale of harmonic oscillators connected end-to-end leads to a string with wave behavior. Moreover, in the continuum limit, the number of degrees of freedom and the lattice spacing are inversely related, so taking the latter to zero is equivalent to taking the former to infinity. Thus, the continuum limit can also be seen as an approximation scheme for simplifying the dynamics of large N systems.

This limit is often applied in solid mechanics to derive dynamical equations for continuous media [AS10] and in statistical physics to study the thermal properties of fields [Kar07]. In such contexts, the dynamical variables are typically measurable quantities that represent the degrees of freedom in the system. However, the continuum limit can more generally be applied to any system where quantities are a function of one independent variable while being related to one another through some recurrence relation.

Outside of physics, recurrence relations are common in combinatorics because counting the elements of a set can often be reduced to a similarly structured counting of elements of a subset. In statistical physics, this combinatorial counting is fundamental to how microstates are summed and partition functions are defined. One might expect that much like more pedestrian combinatorial factors, partition functions should also be governed by recurrence relations. There are simple examples that affirm this expectation. Consider a system with N distinguishable lattice sites and N identical particles. Say that the particles have a unit partition function when free, but when they attach to a lattice site, they gain a Boltzmann factor of $\mathrm{e}^{\beta\lambda}$. The partition function for this system is then

$$Z_N(\beta\lambda) = \sum_{\ell=0}^{N} \binom{N}{\ell} \mathrm{e}^{\beta\lambda\ell}, \tag{2}$$

and from this partition function, it is easy to derive the recurrence equation

$$Z'_N(\beta\lambda) = N\left(Z_N(\beta\lambda) - Z_{N-1}(\beta\lambda)\right). \tag{3}$$

With the initial condition $Z_N(0) = 2^N$ and $Z_0 = 1$, we can solve equation (3) recursively to obtain $Z_N(\beta\lambda)$. Alternatively, we could write equation (3) as the matrix equation

$$\mathbf{Z}'(\beta\lambda) = \hat{M}\mathbf{Z}(\beta\lambda), \tag{4}$$

where $\mathbf{Z}^T = (Z_0(\beta\lambda), Z_1(\beta\lambda), \ldots, Z_N(\beta\lambda))$ and $\hat{M}_{i,j} = i(\delta_{i,j} - \delta_{i,j+1})$. Like with equation (1), it is possible to solve equation (4) using eigendecomposition methods. But again, just like with equation (1), we can introduce lattice spacing between the partition functions for different N and take the system to a continuum limit to derive a new equation that describes the large N system.

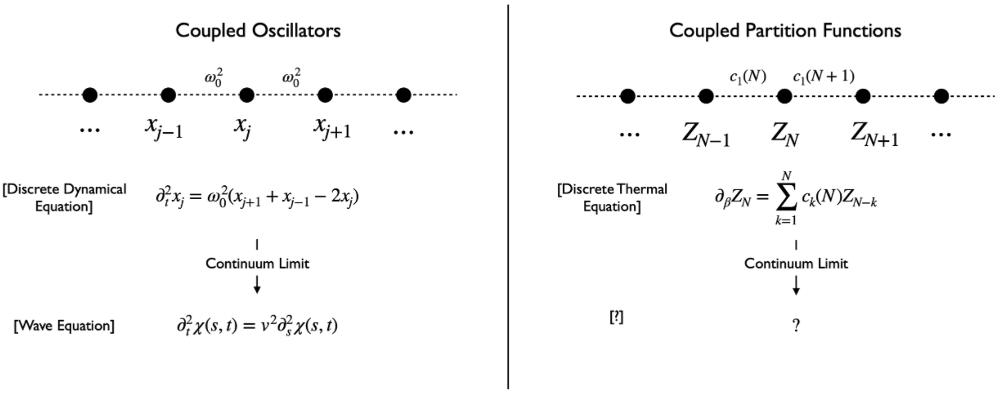

Figure 1. For certain classes of statistical physics models, it is possible to show that partition functions are coupled through a thermal evolution equation. When we take the general form of this equation to the continuum limit, we can derive a partial differential equation that describes the resulting continuum system. This type of limit is common in physics. The left side displays the example of coupled oscillators leading to the wave equation [Pai05], but this example could be replaced with coupled torsional pendulums leading to the sine-Gordon equation [Sco69, DP06], or the master equation for probabilities leading to the Fokker-Planck equation [VK92].

In this paper, we develop the general formalism for making this continuum extrapolation for a class of partition functions and work through four examples that demonstrate that the continuum result has thermal behavior similar to that of the original system. (See figure 1 for a visual motivation of the work).

In section 2, we introduce the class of partition functions amenable to this transformation and show how moments of these partition functions satisfy recurrence relations that become higher-order partial derivatives in the continuum limit. Though it is well known that higher-order moments of a partition function can trivially be expressed as higher-order partial derivatives with respect to the coefficient of the linear term in the energy function, the result here is different. Here we show that higher-order moments become higher-order partial derivatives with respect to the continuum-limit analogue of N.

In section 3, we use the connection between moments and partial derivatives to show that the continuum-limit analogue of the average energy of the partition function yields a partial differential equation. Solving this equation allows us to model the large N properties of the class of partition functions that were defined in section 2.

In sections 4 and 5, we study four examples from this class, reviewing the original partition functions and solving the associated partial differential equation of the continuum limit. We find that both system representations yield similar thermal behavior; in particular, for interacting models, the continuum limit reproduces the functional form of the transition temperature.

In section 6, we discuss what physically these thermal correspondences between discrete and continuous models can provide us in terms of new interpretations of the original model.

2. From moments to derivatives

The continuum limit is often applied in statistical physics to study the thermal properties of fields. In such cases, the variables (e.g. m_i representing the spin at a lattice site i) that define the microstate summation in a partition function are promoted to functions (e.g. $m(\mathbf{x})$ representing the spin-density at a position $\mathbf{x}$) and the associated partition function transitions from a summation over a variable space to an integration over a functional space. However, this paper explores a continuum limit defined by transforming the partition function rather than the variables defining its summation. Ultimately, this transformation gives us a means of computing the large N limit of certain partition functions by solving a partial differential equation rather than summing over Boltzmann-weighted microstates.

First, we state the general class of partition functions to which the transformation applies. Say we have a thermal system whose microstates are defined by the vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$, where $\sigma_i \in \{0,1\}$. We take the energy $\mathcal{E} : \mathbb{R} \to \mathbb{R}$ of the microstate to be a function of the sum of the components of $\boldsymbol{\sigma}$, and each microstate to have a degeneracy factor $\omega : \mathbb{R} \to \mathbb{R}$ that is also only a function of the sum of the components of $\boldsymbol{\sigma}$. The partition function for this thermal system at an inverse temperature β is then

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} \omega\left(\sum\nolimits_i \sigma_i\right) \exp\left[-\beta \mathcal{E}\left(\sum\nolimits_i \sigma_i\right)\right], \tag{5}$$

where $\sum_{\boldsymbol{\sigma}} \equiv \sum_{\sigma_1=0}^{1} \sum_{\sigma_2=0}^{1} \cdots \sum_{\sigma_N=0}^{1}$. We can simplify equation (5) by rewriting it in terms of the macrostate variable $j \equiv \sum_i \sigma_i$

$$Z_N(\beta) = \sum_{j=0}^{N} \binom{N}{j} \omega(j) \, \mathrm{e}^{-\beta \mathcal{E}(j)}. \tag{6}$$

Equation (5) is the starting partition function for our analysis, but we will find it more convenient to work in terms of the equivalent form equation (6). The quantities ω and $\mathrm{e}^{-\beta\mathcal{E}}$ have the same variable argument. Still, they are separated to make it apparent that Z_N is a thermal partition function with both a degeneracy factor and the standard Boltzmann factor. However, for the following derivation, it only matters that Z_N is written as a summation over the binary-valued microstate elements σ_i with summands that are functions of $\sum_i \sigma_i$.

The structure of the general partition function Z_N is central to the subsequent demonstration due to how its moments are related to lower-order partition functions. In general, the subsequent results of apply whenever

1. the microstates in our thermal system can be defined by $\sigma_i = 0$ or 1
2. the degeneracy factor and energy are exclusively functions of $\sum_i \sigma_i$
3. the energy function can be expressed as a finite power series of its argument.

These criteria are generally satisfied for mean-field lattice models [Yeo92].

For the continuum limit explored here, we will find a general way to transform discretely defined moments into partial derivatives. We can understand the general form of this transformation by starting simply. Considering the first moment for Z_N, we have

$$\langle j \rangle_N \equiv \frac{1}{Z_N(\beta)} \sum_{j=0}^{N} \binom{N}{j} j \, \omega(j) \, \mathrm{e}^{-\beta \mathcal{E}(j)}. \tag{7}$$

As was shown in equation (3), we can use Pascal's triangle formula $\binom{N}{j} = \binom{N-1}{j-1} + \binom{N-1}{j}$ to write this moment as

$$\langle j \rangle_N Z_N(\beta) = N Z_N(\beta) - N Z_{N-1}(\beta). \tag{8}$$

From equation (8), we see that the first moment of the partition function can be expressed as the linear combination of two partition functions, with the linear combination looking very much like a finite difference. Performing a similar calculation for the second moment of the partition function yields

$$\langle j^2\rangle_N Z_N(\beta) = N^2 Z_N(\beta) - N(2N-1)Z_{N-1}(\beta) + N(N-1)Z_{N-2}(\beta)\,, \tag{9}$$

which also contains hints of a second-order finite difference. Thus, a two-part pattern is apparent: First, the moments of the partition function Z_N can be written as linear combinations of partition functions Z_ℓ where $\ell \leqslant N$; second, these linear combinations contain expressions found in finite-difference formulas for the same order as the moment.

To proceed, we need to establish the pattern rigorously. Say we want to compute the kth moment of the partition function Z_N. The general kth moment of Z_N is defined as

$$\langle j^k\rangle_N \equiv \frac{1}{Z_N(\beta)}\sum_{j=0}^{N}\binom{N}{j} j^k\,\omega(j)\,\mathrm{e}^{-\beta\mathcal{E}(j)}. \tag{10}$$

Starting from equation (10) and again using Pascal's triangle formula, we can show

$$\langle j^k\rangle_N Z_N(\beta) = N\langle j^{k-1}\rangle_N Z_N(\beta) - N\langle j^{k-1}\rangle_{N-1}Z_{N-1}(\beta)\,. \tag{11}$$

Therefore, the kth moment of the partition function can be written recursively in terms of lower-order moments. The form of equation (11) further suggests that iterating this moment recursion leads to a closed-form expression for $Z_N\langle j^k\rangle_N$ written in terms of a linear combination of Z_ℓ without reference to any specific moments. Taking a generating function approach to solve this recurrence relation (see appendix A), we obtain the expression

$$\langle j^k\rangle_N Z_N(\beta) = \sum_{\ell=0}^{k}(N)_\ell Z_{N-\ell}(\beta)\sum_{r=\ell}^{k}(-1)^r N^{k-r}\binom{k}{r}S(r,\ell) \tag{12}$$

where $(N)_\ell = N(N-1)\cdots(N-\ell+1)$ is the falling factorial, and $S(r,\ell)$ is the Stirling number of the second kind [Weic]. Equation (12) generalizes equations (8) and (9) to arbitrary moments, and it serves as the foundation for our continuum limit.

In any continuum limit transformation, there is a dictionary of mappings for how indices should change to continuous variables and how an index-dependent function should change to one with continuous-variable arguments. Typically, we apply this continuum limit to equations where the index-dependent function appears multiple times in a linear combination. In the case of the continuum limit applied to a chain of oscillators, the continuum limit is applied to the equation of motion of the oscillators. For the case considered here, we will apply the continuum limit to the moment expression equation (12) since the right-hand side of this expression is written as a linear combination of Z_k for various values of k.

For our partition function and its associated arguments, the transformations that define the continuum limit are

$$Z_{N-j}(\beta) \to \mathcal{Z}(\eta - ja;\beta)\ \text{for}\ j\in\mathbb{Z} \quad \text{and} \quad N\to\eta/a. \tag{13}$$

In these transformations, we promote the integer N to a continuous variable η by introducing a lattice spacing a. Where N appears as a factor raised to a power, we transform $N\to\eta/a$, and where N appears within a difference (e.g. $N-2$), we multiply the subtracted quantity by a. In these transformations, we use the new notation $\mathcal{Z}$ to distinguish the transformed partition function from the original Z_N.

To complete the continuum limit, we need to apply the transformations equation (13) to equation (12) and then take the $a\to 0$ limit of the result. Applying the transformations themselves yields a polynomial expression in a, and taking $a\to 0$ ensures that our final transformation is independent of the introduced lattice parameter. Implementing these steps together requires a departure into Stirling numbers and the Egorychev method of proving combinatorial identities (see appendix B), but we ultimately find that the continuum limit of the kth moment is

$$\langle\left(\sum_i \sigma_i\right)^k\rangle_N Z_N(\beta) \longrightarrow \left(\frac{\partial}{\partial \ln \eta}\right)^k \mathcal{Z}(\eta;\beta)\,. \tag{14}$$

Equation (14) states that if Z_N can be expressed as equation (5), then the continuum limit of its kth moments can be expressed as kth order logarithmic derivatives[1] of the partition function with respect to the continuum analogue of N. The next section will show how this transformation can lead to a new way of computing Z_N itself.

3. PDE of continuum partition function

Having established the continuum extrapolation equation (14), we can now obtain the main result for this work. This result will allow us to obtain the large N thermal behavior of a physical system without having to compute the original discrete partition function.

We begin by recalling the structure of the initial partition function:

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} \omega\left(\sum_i \sigma_i\right) \exp\left[-\beta \mathcal{E}\left(\sum_i \sigma_i\right)\right]. \tag{15}$$

The main observable that we can extract from this partition function is the average energy found by taking the β derivative with respect to the partition function:

$$\left\langle \mathcal{E}\left(\sum_i \sigma_i\right)\right\rangle_N Z_N(\beta) = -\frac{\partial}{\partial \beta} Z_N(\beta)\,. \tag{16}$$

Now, let us further constrain our system to have an energy $\mathcal{E}$ that can be expanded as a finite power series in its argument[2]. Namely as,

$$\mathcal{E}(x) = \sum_{k=1}^{M} c_k x^k, \tag{17}$$

where we require M to be a finite integer to ensure that $\mathcal{E}(x)$ is finite for all x. Then equation (16) can be written as

$$\sum_{k=1}^{M} c_k \langle\left(\sum_i \sigma_i\right)^k\rangle_N Z_N(\beta) = -\frac{\partial}{\partial \beta} Z_N(\beta)\,. \tag{18}$$

From here, we use equation (14) to take the continuum limit of this expression. This limit yields the result

$$\sum_{k=1}^{M} c_k \left(\frac{\partial}{\partial \ln \eta}\right)^k \mathcal{Z}(\eta;\beta) = -\frac{\partial}{\partial \beta} \mathcal{Z}(\eta;\beta) \tag{19}$$

[1] The higher order logarithmic derivative on the RHS can be expressed less neatly as $\left(\frac{\partial}{\partial \ln \eta}\right)^k f(\eta) = \left(\eta \frac{\partial}{\partial \eta}\right)^k f(\eta) = (\eta \frac{\partial}{\partial \eta}(\eta \frac{\partial}{\partial \eta} \cdots (\eta \frac{\partial}{\partial \eta} f(\eta)) \cdots))$.

[2] There is nothing in the proof of equation (14) that constrains k to not exceed an arbitrarily large value integer, and thus the energy function need *not* be a finite power series of its argument for the final PDE to be valid continuum limit analogue of the original recurrence relation.. However, constraining the power series to be finite ensures the original energy function is finite for all x and saves us from a more nuanced discussion on the covergence properties of the continuum extrapolation.

or, with the energy definition equation (16), the partial differential equation

$$\left[\frac{\partial}{\partial\beta}+\mathcal{E}\left(\frac{\partial}{\partial\ln\eta}\right)\right]\mathcal{Z}(\eta;\beta)=0. \tag{20}$$

Equation (20) governs the thermal evolution of the continuum limit partition function $\mathcal{Z}(\eta,\beta)$. The interpretation and use of the equation is as follows: For large N, partition functions of the form equation (5) with energy equation (17) can be taken to the continuum limit, and the continuum limit of the partition function 'evolves' in temperature according to equation (20) in much the same way that a system of many discrete oscillators evolves in time according to the wave equation.

Differential equations admit general solutions for their associated functions, and we only obtain specific solutions corresponding to a system of interest by imposing conditions on the independent variables of the equation [Has08]. Thus, from equation (20), we see that we need one condition for β and an unknown number of conditions for η with the number of conditions equivalent to the order of the derivative operator $\mathcal{E}(\partial/\partial\ln\eta)$.

The condition for β can be inferred easily from the original partition function. If the series equation (5) at $\beta=0$ evaluates to a function, then we can define $\Omega(N)\equiv Z_N(\beta=0)$ and take

$$\mathcal{Z}(\eta;\beta=0)=\Omega(\eta). \tag{21}$$

For η, there are no natural conditions on $\mathcal{Z}$ at any particular value of η, and the value of $\mathcal{Z}$ at $\eta=0$ or $\eta=\infty$ is determined by $\Omega(\eta)$ and the structure of the general solution to equation (20). As an additional constraint on this class of partition functions, we require that the partition function not diverge at zero temperature. This ensures that Z remains an analytical function for all temperatures, and thus, the associated partial differential equation can yield finite solutions for the $\beta\to\infty$ limit. This condition of non-divergence of the partition function manifests as a constraint on the energy function: We must have $\mathcal{E}(x)\geqslant 0$ for all x.

4. Linear energy

In the following two sections, we show that the continuum limit of a partition function with a linear energy yields thermal behavior similar to that of the partition function of the original system. We will pay particular attention to transition temperatures and show that the continuum limit reproduces the parameter-scaling behavior of these temperatures.

In this section, we apply equation (20) to two models, both of which have energies of the form

$$\mathcal{E}\left(\sum\sigma_i\right)=\lambda\sum_{i=1}^{N}\sigma_i \tag{22}$$

where $\sigma_i\in\{0,1\}$ and $\lambda>0$ has units of energy. Typically, linear energies like equation (22) yield partition functions that are simple enough to compute directly, but here we will explore how the continuum limit can recapture some of the results of the straightforward calculation.

From equation (20), we see that the partial differential equation obtained from the continuum limit of a system with energy equation (22) is

$$\left(\frac{\partial}{\partial\beta}+\lambda\frac{\partial}{\partial\ln\eta}\right)\mathcal{Z}(\eta;\beta)=0. \tag{23}$$

This is a first-order PDE in two variables, and as such, it has a simple general solution. As can be easily checked, the PDE $\partial_x f(x,y)+\partial_y f(x,y)=0$ has the general solution $f(x,y)=u(x-y)$

given the condition $u(x) \equiv f(x, y = 0)$ for a real-valued function $u(s)$. Given $\mathcal{Z}(\eta; \beta = 0) = \Omega(\mathrm{e}^{\ln \eta})$, we then find that the specific solution is

$$\mathcal{Z}(\eta; \beta) = \Omega\left(\mathrm{e}^{\ln \eta - \beta\lambda}\right) = \Omega\left(\eta \mathrm{e}^{-\beta\lambda}\right). \tag{24}$$

We will use equation (24) in what follows to find the continuum limit partition function for two models.

4.1. *Repulsive lattice-gas model*

Of the non-trivial systems whose partition functions have the form equation (5), the lattice-gas model considered in equation (2) is the simplest. In the lattice-gas model, we have a system of N particles and N lattice sites. Each lattice site i can either be empty ($\sigma_i = 0$) or occupied ($\sigma_i = 1$) by a particle. For the repulsive model, we assume the particles are repelled by the lattice sites, and thus, there is an energy cost to a site being occupied by a particle. Specifically, the system's total energy increases by $\lambda > 0$ for each occupied site.

4.1.1. Original model. From this setup, and ignoring the free partition functions for the individual particles, the partition function for the entire system is

$$Z_{\text{lattice-gas}} = \sum_{\boldsymbol{\sigma}} \exp\left(-\beta\lambda \sum_{i=1}^{N} \sigma_i\right) = \left(1 + \mathrm{e}^{-\beta\lambda}\right)^N. \tag{25}$$

We must have $\lambda > 0$ in this system for the partition function to be finite for all $T \in [0, \infty)$. From equation (25), we can compute the average number of occupied lattice sites at a constant temperature. Denoting this average occupancy as $\langle \ell \rangle$, we have

$$\langle \ell \rangle \equiv -\frac{\partial}{\partial(\beta\lambda)} \ln Z_{\text{lattice-gas}} = \frac{N\mathrm{e}^{-\beta\lambda}}{1 + \mathrm{e}^{-\beta\lambda}}, \tag{26}$$

and the temperature-limit behavior of equation (26) is

$$\lim_{T \to 0} \langle \ell \rangle = 0, \qquad \lim_{T \to \infty} \langle \ell \rangle = N/2 < N. \tag{27}$$

Qualitatively, equation (27) reflects that lowering the temperature for this repulsive-lattice system leads all particles to leave the lattice while increasing the temperature to infinity makes the repulsive properties irrelevant and site occupancy completely random. Does the continuum limit of this system give the same thermal behavior for the appropriate analogue of $\langle \ell \rangle$?

4.1.2. Continuum limit. Equation (24) immediately gives us the solution for the continuum limit of the linear energy system, provided we know the counting of the microstates at zero temperature. For the lattice-gas model, we have two states for each lattice site, and thus $\Omega(N) = 2^N$ or, in terms of the continuum variable η,

$$\Omega(\eta) = 2^\eta. \tag{28}$$

Therefore, with equation (24), we find that the solution to equation (23) is

$$\mathcal{Z}_{\text{lattice-gas}}(\eta; \beta) = \exp\left(\eta \mathrm{e}^{-\beta\lambda} \ln 2\right). \tag{29}$$

If we define the continuum analogue of the average occupancy in a way similar to the definition in equation (26), we find

$$\ell_{\text{avg}} \equiv -\frac{\partial}{\partial(\beta\lambda)} \ln \mathcal{Z}_{\text{lattice-gas}}(\eta; \beta) = \eta \mathrm{e}^{-\beta\lambda} \ln 2. \tag{30}$$

Now considering the limit behavior of equation (30) we have

$$\lim_{T\to 0}\ell_{\text{avg}} = 0, \qquad \lim_{T\to\infty}\ell_{\text{avg}} = \eta\ln 2 < \eta, \tag{31}$$

which reproduces the qualitative low-temperature behavior of the original result. Namely, at zero temperature, the average occupancy is zero. At infinite temperature, the system's entropy forces the average occupancy to scale with η, although the exact functional form of that scaling is different from that in equation (27).

Despite the similarities in thermal limits, the order parameter of the original system equation (26) and the associated continuum limit version equation (30) do not have similar functional forms. This difference reveals a limitation we will see again throughout our examples of the continuum limit: Though thermal properties related to transitions can be reproduced, the functional forms of order parameters are not generally reproduced. The coarse-graining associated with taking the partition function to the continuum limit seems to eliminate the specific functional behavior of observables while preserving the thermal transitions that define them.

The model considered in this section had no such thermal transitions and was thus fairly simple. In the next section, we explore another linear model with a thermal transition that is present and easy to compute.

4.2. Permutation model

Williams [Wil17] introduced a statistical physics model whose state space consisted of permutations of a list and whose energy was a function of the number of derangements of that list. The initial partition function for this system can be written as a sum over the set of microstates $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ where $\sigma_i \in \{0,1\}$, that is as

$$Z_{\text{symm.}}(\beta\lambda) = \sum_{\boldsymbol{\sigma}} \mathrm{d}\left(\sum_{i=1}^{N}\sigma_i\right)\exp\left(-\beta\lambda\sum_{i=1}^{N}\sigma_i\right), \tag{32}$$

and where $d(n)$ is the number of derangements of an n element list.

4.2.1. Original model. In [Wil17] it was found that the integral definition of $d(n)$ reduced equation (32) to

$$Z_{\text{symm.}}(\beta\lambda) = \int_0^{\infty}\mathrm{d}x\,\mathrm{e}^{-x}\left(1+(x-1)\,\mathrm{e}^{-\beta\lambda}\right)^N, \tag{33}$$

and that the average number of deranged elements (i.e. the elements not in their correct position, denoted $\langle j\rangle$) had the large N limit

$$\langle j\rangle \equiv -\frac{\partial}{\partial(\beta\lambda)}\ln Z_{\text{symm.}} \simeq N - \mathrm{e}^{\beta\lambda}. \tag{34}$$

Next, with the requirement that there can never be fewer than 0 deranged elements, it was concluded that the system had a transition temperature

$$k_{\mathrm{B}}T_0 = \frac{\lambda}{\ln N}, \tag{35}$$

which was later verified by simulations.

4.2.2. Continuum limit. Can the temperature equation (35) be reproduced using the continuum-limit formalism? To answer this, we start from equation (24), the general solution to the linear PDE of the continuum limit. This solution requires us to find $\Omega(N)$ defined by $Z_N(\beta=0)=\Omega(N)$. From equation (33), we find $Z(\beta=0)_{\text{symm.}}=\Gamma(N+1)$ where $\Gamma(x)$ is the gamma function. Thus we have

$$\Omega(\eta)=\Gamma(\eta+1). \tag{36}$$

Therefore, by equation (24), the solution to the continuum extrapolation of the system is

$$\mathcal{Z}_{\text{symm.}}(\eta;\beta)=\Gamma\left(\eta\mathrm{e}^{-\beta\lambda}+1\right). \tag{37}$$

To determine whether this result embodies thermal behavior similar to that of equation (34), we need to find an analogous expression for $\langle j\rangle$, the average number of deranged elements. We call this analogous quantity j_{avg}. Using the definition equation (34) as a motivation, we can compute j_{avg} as

$$j_{\text{avg}}\equiv-\frac{\partial}{\partial(\beta\lambda)}\ln\mathcal{Z}_{\text{symm.}}(\eta;\beta)=\eta\mathrm{e}^{-\beta\lambda}\psi_0\left(\eta\mathrm{e}^{-\beta\lambda}+1\right), \tag{38}$$

where $\psi_0(x)\equiv\Gamma'(x)/\Gamma(x)$ is the digamma function. The transition temperature equation (35) was derived by imposing $\langle j\rangle\geqslant 0$ on equation (34). Similarly, we can require $j_{\text{avg}}\geqslant 0$. Given that $\psi_0(x)$ is a monotonically increasing function for $x\geqslant 0$ and that it has a root at $x_0\simeq 1.461\,63\equiv 1+\epsilon$ [Weia], we find that imposing $j_{\text{avg}}\geqslant 0$ on equation (38) implies that $\eta\mathrm{e}^{-\beta\lambda}+1\geqslant 1+\epsilon$ or, equivalently, $k_\mathrm{B}T\geqslant k_\mathrm{B}\widetilde{T}_0$ where we defined

$$k_\mathrm{B}\widetilde{T}_0=\frac{\lambda}{\ln(\eta/\epsilon)}. \tag{39}$$

The variable η is the continuous analogue of the parameter N, so we see that equation (39) has a similar functional dependence on η as equation (35) has on N. Thus, the continuum limit reproduces the scaling dependence of the transition temperature.

5. Quadratic energy

In the previous section, we applied equation (20) to systems whose energy is a linear function of its argument. Here, we consider systems where the energy is a quadratic function of its argument. For such systems, the energy, as a function of $\sum_{i=1}^N\sigma_i$, takes the form

$$\mathcal{E}\left(\sum\sigma_i\right)=\lambda_1\sum_{i=1}^N\sigma_i+\frac{\lambda_2}{2}\sum_{i,j=1}^N\sigma_i\sigma_j, \tag{40}$$

where $\sigma_i\in\{0,1\}$ and λ_1 and λ_2 have units of energy. By the result given in section 3, the continuum extrapolation of the system with energy equation (40) is

$$\left(\frac{\partial}{\partial\beta}+\lambda_1\frac{\partial}{\partial\ln\eta}+\frac{\lambda_2}{2}\frac{\partial^2}{\partial\ln\eta^2}\right)\mathcal{Z}(\eta;\beta)=0, \tag{41}$$

with the initial condition $\mathcal{Z}(\eta;\beta=0)=\Omega(\eta)$.

Equation (41) is a second-order differential equation with the form of the heat equation. However, in order to use the heat equation solution, we need the coefficient λ_2 to be less than zero. Otherwise, we would have to contend with an ill-posed inverse problem [Isa06, Kab08]. So, for simplicity, we set

$$\lambda_2=-|\lambda_2|. \tag{42}$$

For equations of the form equation (41), we can write the general solution using the Green's function solution of the heat equation (chapter 20 of [Has13]). Defining the Green's function $\mathcal{G}(\ln\eta, \ln\eta'; \beta)$ implicitly through

$$\left(\frac{\partial}{\partial\beta} + \lambda_1 \frac{\partial}{\partial \ln\eta} - \frac{|\lambda_2|}{2}\frac{\partial^2}{\partial \ln\eta^2}\right)\mathcal{G}(\ln\eta, \ln\eta'; \beta) = \delta(\ln\eta - \ln\eta'), \tag{43}$$

where $\delta(x)$ is the Dirac Delta function, and where $\mathcal{G}$ has the boundary conditions $\lim_{\beta\to 0}\mathcal{G}(x, x'; \beta) = \delta(x - x')$ and $\lim_{x\pm\infty}\mathcal{G}(x, x'; \beta) = 0$, we find that the general solution to equation (41) is

$$\mathcal{Z}(\eta;\beta) = \int_{-\infty}^{\infty} \mathrm{d}\ln\eta'\, \mathcal{Z}(\eta';0)\, \mathcal{G}(\ln\eta, \ln\eta'; \beta). \tag{44}$$

The Green's function for equation (43) is that for the heat equation:

$$\mathcal{G}(\ln\eta, \ln\eta'; \beta) = \frac{1}{\sqrt{2\pi\beta|\lambda_2|}}\exp\left[-\frac{1}{2\beta|\lambda_2|}(\ln\eta - \beta\lambda_1 - \ln\eta')^2\right]. \tag{45}$$

Inserting equation (45) into equation (44) with the initial condition $\mathcal{Z}(\eta;0) = \Omega(\eta)$, we obtain

$$\mathcal{Z}(\eta;\beta) = \frac{1}{\sqrt{2\pi\beta|\lambda_2|}}\int_{-\infty}^{\infty}\mathrm{d}\ln\eta'\,\Omega(\eta')\exp\left[-\frac{1}{2\beta|\lambda_2|}\ln^2\left(\eta \mathrm{e}^{-\beta\lambda_1}/\eta'\right)\right]. \tag{46}$$

An alternative derivation of equation (46) is given in appendix D.

Unless $\Omega(\eta)$ has a simple exponential dependence, integrals of the form equation (46) are not computable in closed-form, so to explore the properties of the continuum limit solutions to equation (41), we will apply an integral approximation. We write equation (46) as

$$\mathcal{Z}(\eta;\beta) = \int_{-\infty}^{\infty}\mathrm{d}\ln\eta'\,\exp\left[-F(\ln\eta'; \ln\eta, \beta)\right], \tag{47}$$

where we defined

$$F(\ln\eta'; \ln\eta, \beta) \equiv -\ln\Omega(\eta') + \frac{1}{2\beta|\lambda_2|}\ln^2\left(\eta\mathrm{e}^{-\beta\lambda_1}/\eta'\right) + \frac{1}{2}\ln(2\pi\beta|\lambda_2|). \tag{48}$$

Applying Laplace's method to equation (47), we have

$$\mathcal{Z}(\eta;\beta) \simeq \frac{\exp\left[-F(\ln\bar{\eta}; \ln\eta, \beta)\right]}{\sqrt{2\pi F''(\ln\eta'; \ln\eta, \beta)\,|_{\eta=\bar{\eta}}}}, \tag{49}$$

where $\bar{\eta}$ is implicitly defined by the condition

$$0 = F'(\ln\eta'; \ln\eta, \beta)|_{\eta=\bar{\eta}} = -\partial_{\ln\eta'}\ln\Omega(\eta')\,|_{\eta=\bar{\eta}} + \frac{1}{\beta|\lambda_2|}\ln\left(\bar{\eta}\mathrm{e}^{\beta\lambda_1}/\eta\right), \tag{50}$$

and the inequality

$$F''(\ln\eta'; \ln\eta, \beta)\,|_{\eta=\bar{\eta}} = -\partial^2_{\ln\eta'}\ln\Omega(\eta') + \frac{1}{\beta|\lambda_2|} > 0. \tag{51}$$

To connect $\bar{\eta}$ to a physical observable, we can compute the order parameter from the saddle-point free energy equation (48). For a partition function Z with energy equation (40), the thermal average of $j = \sum_{i=1}^{N}\sigma_i$ is

$$\langle j\rangle = -\frac{\partial}{\partial(\beta\lambda_1)}\ln Z_N(\beta). \tag{52}$$

Making the continuum limit transformation, we assert that the thermal average becomes $\langle j \rangle \to j_{\text{avg}}$ where

$$j_{\text{avg}} = -\frac{\partial}{\partial(\beta\lambda_1)} \ln \mathcal{Z}(\eta;\beta). \tag{53}$$

With the integral approximation equation (49), we in turn approximate j_{avg} as

$$j_{\text{avg}} \simeq \frac{\partial}{\partial(\beta\lambda_1)} F(\ln\bar{\eta};\ln\eta,\beta). \tag{54}$$

Using equations (48) and (50), we then obtain

$$j_{\text{avg}} \simeq \partial_{\ln\eta'} \ln\Omega(\eta')\,|_{\eta=\bar{\eta}}. \tag{55}$$

Therefore, we can write the continuum analogue of the thermal average of $\sum_{i=1}^{N}\sigma_i$ in terms of the critical point $\bar{\eta}$.

With equations (50) and (51) as the derived constraints on $\bar{\eta}$, we can now consider the interacting analogue of the first system considered in section 4. We will focus on the appearance of phase transitions and the parameter dependencies of the temperatures that define them.

5.1. Interacting repulsive lattice-gas model

Having developed the general formalism for the continuum limit of quadratically interacting mean-field models, we can revisit the lattice gas model of section 4.1, but now include global interactions between lattice sites. Upon including the global interactions of the form equation (40) (with the specification $\lambda_2 = -|\lambda_2|$), the system partition function becomes

$$Z_{\text{int-lattice-gas}} = \sum_{\boldsymbol{\sigma}} \exp\left(-\beta\lambda_1 \sum_{i=1}^{N}\sigma_i + \frac{1}{2}\beta|\lambda_2| \sum_{i,j=1}^{N}\sigma_i\sigma_j\right). \tag{56}$$

Our objective is to determine the thermal properties of the system defined by equation (56) and then compare these properties with what we find by solving the continuum extrapolation of this system.

5.1.1. Original model.

First, we review one approach to obtaining the thermal properties of equation (56). Using the Hubbard–Stratonovich transformation [Hub59], we can rewrite the partition function as

$$Z_{\text{int-lattice-gas}} = \int_{-\infty}^{\infty} \mathrm{d}x \exp\left[-F_{\text{int-lattice-gas}}(x)\right], \tag{57}$$

where we defined

$$F_{\text{int-lattice-gas}}(x) \equiv \frac{x^2}{2\beta|\lambda_2|} - N\ln\left(\mathrm{e}^{x-\beta\lambda_1}+1\right) + \frac{1}{2}\ln\left(2\pi\beta|\lambda_2|\right). \tag{58}$$

We then approximate equation (57) using Laplace's method in the $N \gg 1$ limit. Doing so requires us to determine the value $x = \bar{x}$ at which the integrand is maximized. From equation (57), the two conditions that define this $\bar{x}$ are the critical point condition

$$\partial_x F|_{x=\bar{x}} = \frac{\bar{x}}{\beta|\lambda_2|} - \frac{N\mathrm{e}^{\bar{x}-\beta\lambda_1}}{\mathrm{e}^{\bar{x}-\beta\lambda_1}+1} = 0, \tag{59}$$

and the stability condition

$$\partial_x^2 F|_{x=\bar{x}} = \frac{1}{\beta|\lambda_2|} - \frac{N\mathrm{e}^{\bar{x}-\beta\lambda_1}}{\left(\mathrm{e}^{\bar{x}-\beta\lambda_1}+1\right)^2} > 0. \tag{60}$$

Using equation (59) in equation (60), we obtain

$$\partial_x^2 F|_{x=\bar{x}} = \frac{1}{\beta|\lambda_2|}\left[1-\bar{x}\left(1-\frac{\bar{x}}{\beta|\lambda_2|N}\right)\right] > 0. \tag{61}$$

It is possible to show that equation (61) is always true if $\beta|\lambda_2|N < 4$, i.e. if the quadratic function of $\bar{x}$ within the brackets has no real roots. In such a case, the solution to equation (59) is always stable. However, if the quadratic function does have real roots, then the solution is stable if it satisfies the inequality equation (61). This mathematical transition from stability always being true to stability being conditionally true parallels a physical phase transition in the system. Thus we conclude that $\bar{x}$ is always a stable solution for $\beta < \beta_c$ where β_c, the critical inverse-temperature, is given by

$$\beta_c \equiv \frac{4}{N|\lambda_2|}. \tag{62}$$

5.1.2. Continuum limit. Having obtained the critical temperature for the original system, we next consider whether the continuum extrapolation of this system can lead to a similar transition temperature. We determine this by considering the equations equations (50) and (51) for the appropriate $\Omega(\eta)$. In section 4.1, we found $\Omega(\eta) = 2^\eta$ for the lattice-gas system. Thus, we have $\ln\Omega(\eta) = \mathrm{e}^{\ln\eta}\ln 2$, and equations (50) and (51) give us the conditions

$$0 = \partial_{\ln\eta'} F(\ln\eta';\ln\eta,\beta)|_{\eta=\bar{\eta}} = -\mathrm{e}^{\ln\bar{\eta}}\ln 2 + \frac{1}{\beta|\lambda_2|}\ln\left(\bar{\eta}\mathrm{e}^{\beta\lambda_1}/\eta\right) \tag{63}$$

$$\partial^2_{\ln\eta'} F(\ln\eta';\ln\eta,\beta)|_{\eta=\bar{\eta}} = -\mathrm{e}^{\ln\bar{\eta}}\ln 2 + \frac{1}{\beta|\lambda_2|} > 0. \tag{64}$$

Solving equation (63) yields

$$\ln\bar{\eta} = \ln\left(\eta\mathrm{e}^{-\beta\lambda_1}\right) - W\left(-\beta|\lambda_2|\ln 2\,\eta\mathrm{e}^{-\beta\lambda_1}\right), \tag{65}$$

where W is the Lambert W function. Substituting the condition equation (63) into equation (64), we find that equation (65) is stable if $1 > \ln\bar{\eta} - \ln(\eta\mathrm{e}^{-\beta\lambda_1})$ or, equivalently, if $1 > -W(-\beta|\lambda_2|\ln 2\,\eta\mathrm{e}^{-\beta\lambda_1})$. This inequality is true if $W = W_0$ (i.e. if W is the principal branch of the Lambert W function) and if W_0 is a real quantity. A fundamental property of the Lambert W function is that it is real if and only if the argument of W is greater than or equal to $-\mathrm{e}^{-1}$. If the argument of W in equation (65) falls below this threshold, then the solution is not real, equation (63) doesn't yield a physical quantity, and the saddle-point approximation no longer applies. Thus in order for equation (65) to be a valid stable solution, the system parameters must satisfy

$$-\mathrm{e}^{-1} \leqslant -\beta|\lambda_2|\ln 2\,\eta\mathrm{e}^{-\beta\lambda_1}. \tag{66}$$

The inverse-temperature β is the only parameter we can modulate in a thermal system, so we must solve for the condition on β that allows equation (66) to be satisfied. Again, using the Lambert W function to find this condition, we find that equation (66) is satisfied if β in turn satisfies

$$\beta \leqslant \beta_c = -\frac{1}{\lambda_1}W_0\left(-\frac{\lambda_1}{\eta|\lambda_2|\mathrm{e}\ln 2}\right), \tag{67}$$

where we chose the principal branch to ensure that the inequality remained well-defined for small arguments of the Lambert W function[3]. Considering equation (67) for $\eta \gg 1$ (as an analogue to our $N \gg 1$ consideration) and assuming λ_1 and $|\lambda_2|$ are of the same order of magnitude, we can approximate equation (67) with $W(x) = x + O(x^2)$. With this approximation, we find that equation (65) is stable if $\beta \leqslant \beta_c$, where the critical inverse-temperature is

$$\beta_c = \frac{1}{\eta|\lambda_2|\mathrm{e}\ln 2} + O\left(\eta^{-2}\right). \tag{68}$$

Comparing this result to the corresponding result for the original statistical physics system, we see that the $\sim 1/N|\lambda_2|$ dependence of equation (62) is matched by the $\sim 1/\eta|\lambda_2|$ dependence of equation (68). Thus, the continuum limit of the original system yields a transition temperature with the same parameter scaling behavior as that of the original system.

5.2. *Interacting permutation model*

Similar to the extension of the previous section, we can extend our system in section 4.2 to one involving interactions between permuted elements. This extension was presented in [Wil17], so we just review the major results here.

5.2.1. Original model. The state space of our system consists of permutations of a list, and for the case of a mean-field interaction model, we take the system's energy to be a quadratic function of the number of derangements in the list. For an energy function $\mathcal{H}_{\text{int-symm}} = \beta\lambda_1 j - \beta|\lambda_2|j^2/2$, we find that the partition function for the system can be written as

$$Z_{\text{int-symm.}} = \sum_{j=0}^{N} \binom{N}{j} \mathrm{d}(j) \exp\left(-\beta\lambda_1 j + \frac{\beta|\lambda_2|}{2} j^2\right) \equiv \sum_{j=0}^{N} \mathrm{e}^{-\beta f_N(j;\lambda_1,|\lambda_2|,\beta)}, \tag{69}$$

where we defined the free energy function f_N as

$$f_N(j,\lambda_1,|\lambda_2|,\beta) \equiv \lambda_1 j - \frac{|\lambda_2|}{2} j^2 - \frac{1}{\beta}\ln\left[\binom{N}{j} d_j\right]. \tag{70}$$

For sufficiently large j, we have the approximation $d(j) \simeq j!/e$. Thus, using the Gamma function expression of the factorial ($n! = \Gamma(n+1)$), we obtain

$$f_N(j,\lambda_1,|\lambda_2|,\beta) = \lambda_1 j - \frac{|\lambda_2|}{2} j^2 + \frac{1}{\beta}\ln\Gamma(N-j+1) + f_0, \tag{71}$$

where f_0 represents constants and sub-leading corrections in j. Taking equation (71) to define the Landau free energy of the system, we can use it to compute the order parameter along with the critical temperatures that constrain the system's phase behavior. Imposing the critical point condition $\partial_j f_N(j,\lambda_1,|\lambda_2|,\beta)|_{j=\bar{j}} = 0$ yields

$$\partial_j f_N(j,\lambda_1,|\lambda_2|,\beta)|_{j=\bar{j}} = \lambda_1 - |\lambda_2|\bar{j} - \frac{1}{\beta}\psi^{(0)}(N-\bar{j}+1) = 0 \tag{72}$$

[3] By the properties of the Lambert W function, we also require $-e^{-1} < -\lambda_1/(\eta|\lambda_2|e\ln 2)$, but given our $\eta \gg 1$ limit, we can assume that this criterion is automatically satisfied.

where $\psi^{(0)}(x)$ is the digamma function. Further imposing the stability condition $\partial_j^2 f_N(j,\lambda_1,|\lambda_2|,\beta)|_{j=\bar{j}} > 0$ yields

$$\partial_j^2 f_N(j,\lambda_1,|\lambda_2|,\beta)|_{j=\bar{j}} = -|\lambda_2| + \frac{1}{\beta}\psi^{(1)}(N-\bar{j}+1) > 0, \tag{73}$$

where $\psi^{(1)}(x) \equiv \mathrm{d}\psi^{(0)}(x)/\mathrm{d}x$. To simplify these expressions, we use the digamma function approximation $\psi^{(0)}(x) = \ln x - 1/2x + O(x^{-2})$. With this approximation, equation (72), becomes

$$\mathrm{e}^{-\beta|\lambda_2|\bar{j}} = -\mathrm{e}^{-\beta\lambda_1}(j-N) \tag{74}$$

which can be solved to yield

$$\bar{j} = N + \frac{1}{\beta|\lambda_2|}W\left(-\beta|\lambda_2|\mathrm{e}^{\beta\lambda_1-\beta|\lambda_2|N}\right), \tag{75}$$

where W is the Lambert W function with a currently unspecified branch. Two conditions constrain this solution. First, it must be greater than or equal to zero: $\bar{j} \geqslant 0$. Second, it must represent a stable value of Landau free energy: $\partial_j^2 f_N|_{j=\bar{j}} > 0$.

For the first condition, we find that equation (75) satisfies $\bar{j} \geqslant 0$ when the system parameters satisfy

$$1 \geqslant -\frac{1}{\beta|\lambda_2|N}W\left(-\beta|\lambda_2|N\mathrm{e}^{-\beta|\lambda_2|N}\frac{\mathrm{e}^{\beta\lambda_1}}{N}\right). \tag{76}$$

We multiplied and divided the expression in the argument by N for later convenience. Since the Lambert W function is a monotonically increasing function that is defined by $x = W(x\mathrm{e}^x)$, it has the property that $W(x\mathrm{e}^x a)/x \leqslant 1$ if and only if $a \leqslant 1$. Thus equation (76) implies that $\bar{j} \geqslant 0$ is satisfiable if and only if

$$\frac{\mathrm{e}^{\beta\lambda_1}}{N} \leqslant 1. \tag{77}$$

The boundary of this inequality gives us the first critical temperature of the system:

$$\beta_{c1}\lambda_1 = \ln N \qquad [\bar{j} \geqslant 0\text{-condition temperature}]. \tag{78}$$

In order to have equation (75) satisfy $j > 0$, we require $\beta < \beta_{c1}$.

For the second condition, we apply $\partial_j^2 f_N|_{j=\bar{j}} > 0$ to equation (73). Using the approximation $\psi^{(1)}(x) = 1/x + O(x^{-2})$, we obtain

$$\begin{aligned}\partial_j^2 f_N(j,\lambda_1,|\lambda_2|,\beta)|_{j=\bar{j}} &= -|\lambda_2| + \frac{1}{\beta}\frac{1}{N-\bar{j}} \\ &= -|\lambda_2|\left(1 + \frac{1}{W\left(-\beta|\lambda_2|\mathrm{e}^{\beta\lambda_1-\beta|\lambda_2|N}\right)}\right),\end{aligned} \tag{79}$$

where in the second line, we used equation (75). Equation (79) is positive (thus suggesting a stable critical point) if

$$-1 < W\left(-\beta|\lambda_2|\mathrm{e}^{\beta\lambda_1-\beta|\lambda_2|N}\right) < 0. \tag{80}$$

The only branch whose range of values can satisfy equation (80) is the principal branch $W_0(x)$. Therefore, the Lambert W function in equation (75) is the principal-branch function. The right inequality in equation (80) is automatically satisfied given the negative argument of the

Lambert W function. The left inequality is satisfied if and only if the argument of the Lambert W function is greater than $-\mathrm{e}^{-1}$. This constraint gives us the condition

$$-\mathrm{e}^{-1} < -\beta|\lambda_2|\mathrm{e}^{\beta\lambda_1-\beta|\lambda_2|N}. \tag{81}$$

Solving for the temperature that satisfies this condition gives us

$$\beta(\lambda_1-|\lambda_2|N) < W\left(\frac{\mathrm{e}^{-1}}{|\lambda_2|}(\lambda_1-|\lambda_2|N)\right). \tag{82}$$

Considering the boundary of the inequality in equation (82) gives us the second critical temperature for the system:

$$\beta_{c2} = \frac{1}{\lambda_1-|\lambda_2|N}W\left(\frac{\mathrm{e}^{-1}}{|\lambda_2|}(\lambda_1-|\lambda_2|N)\right) \qquad \text{[Stability-condition temperature]}. \tag{83}$$

The inequality $\beta<\beta_{c2}$ defines a necessary and sufficient condition for equation (75) to be a stable equilibrium of the Landau free energy. However, the properties of the Lambert W function allow us to find an accompanying necessary but not sufficient (NBNS) condition that β must satisfy for the system to have a stable equilibrium at equation (75). This condition will be useful when we consider the continuum limit of this system later in this section. From the identity $W(x)/x=\mathrm{e}^{-W(x)}$, we can show that $W(Bx)/x<B$ and thus that $A<W(Bx)/x$ implies $A<B$. Thus, we find that a necessary (but not sufficient condition) that β must satisfy for equation (75) to be a stable equilibrium is $\beta<\beta_0$, where

$$\beta_0 = \frac{\mathrm{e}^{-1}}{|\lambda_2|} \qquad \text{[NBNS stability-condition temperature]}. \tag{84}$$

Equations (78) and (83) (and relatedly equation (84)) represent the two main temperatures that define the phase behavior of the mean-field interacting permutation system. Now, we consider the continuum limit analogue of this system to see if the salient aspects of the thermal behavior are reproduced.

5.2.2. Continuum limit. First, we recall that the microstate function $\Omega(\eta)$ from the linear-energy permutation model is $\Omega(\eta)=\Gamma(\eta+1)$. The critical point conditions equations (50) and (51) thus become

$$0=\partial_{\ln\eta'}F(\ln\eta';\ln\eta,\beta)|_{\eta=\bar\eta} = -\bar\eta\psi^{(0)}(\bar\eta+1)+\frac{1}{\beta|\lambda_2|}\ln\left(\bar\eta\mathrm{e}^{\beta\lambda_1}/\eta\right) \tag{85}$$

$$\partial^2_{\ln\eta'}F(\ln\eta';\ln\eta,\beta)|_{\eta=\bar\eta} = -\bar\eta\psi^{(0)}(\bar\eta+1)-\bar\eta^2\psi^{(1)}(\bar\eta+1)+\frac{1}{\beta|\lambda_2|}>0, \tag{86}$$

where $\psi^{(0)}(x)$ is the digamma function and $\psi^{(1)}(x)$ is its derivative. To make progress, we approximate $\psi^{(0)}(x)\simeq\ln x-O(x^{-1})$ for large x in both equations (85) and (86). Inserting this approximation into equation (85), we obtain the self-consistency equation

$$\ln\bar\eta = \frac{\ln\eta-\beta\lambda_1}{1-\beta|\lambda_2|\bar\eta}+O\left(\bar\eta^{-1}\right) \tag{87}$$

and using the approximation $\psi^{(1)}(x)\simeq x^{-1}-O(x^{-2})$ we find that equation (86) yields the stability condition

$$\bar\eta(\ln\bar\eta+1)+O(1)<\frac{1}{\beta|\lambda_2|}, \tag{88}$$

or equivalently

$$\ln\bar{\eta} < W_0\left(\frac{\mathrm{e}}{\beta|\lambda_2|}\right) - 1. \tag{89}$$

Now with equation (55), we find that the order parameter of the system has the form

$$j_{\text{avg}} \simeq \bar{\eta}\psi^{(0)}\left(\bar{\eta}+1\right) = \bar{\eta}\ln\bar{\eta} + O\left(1\right). \tag{90}$$

Thus, to leading order, we can take $j_{\text{avg}} = \bar{\eta}\ln\bar{\eta}$. In the original thermal system, $\langle j\rangle = \sum_{i=1}^{N}\langle\sigma_i\rangle \geqslant 0$. Assuming the continuum extrapolation also obeys this inequality, we can infer from equation (90) that $\bar{\eta} \geqslant 1$ and thus $\ln\bar{\eta} \geqslant 0$. Considering equation (87) in the context of this inequality, we conclude that the numerator and the denominator must both be positive quantities[4]. Requiring the numerator to be positive gives $\ln\eta \geqslant \beta\lambda_1$, which in turn yields the first critical temperature condition for the system:

$$\tilde{\beta}_{c1}\lambda_1 = \ln\eta \qquad [j_{\text{avg}} \geqslant 0\text{-condition temperature}]. \tag{91}$$

Equation (91) matches the form of the $\langle j\rangle \geqslant 0$ condition equation (78). Requiring the denominator of equation (87) to be positive we find $1 - \beta|\lambda_2|\bar{\eta} > 0$ or $\beta < 1/|\lambda_2|\bar{\eta}$ which cannot be solved for β since $\bar{\eta}$ is implicitly a function of β. Instead, using the fact that $\bar{\eta} \geqslant 1$, we find that a NBNS condition for the denominator of equation (91) to be non-negative is $\beta < 1/|\lambda_2|$. Inspecting equation (89) shows that this same inequality is also a NBNS condition for the point $\bar{\eta}$ to define a stable critical point of the system. Thus we can conclude that the condition $\beta < \tilde{\beta}_0$, where

$$\tilde{\beta}_0 = \frac{1}{|\lambda_2|} \qquad [\text{NBNS stability-condition temperature}], \tag{92}$$

defines a NBNS condition for the stability of the system. This condition matches the scaling of the analogous NBNS condition equation (84) for the original system. Of course, it would have been preferred to find some way to solve equation (89) for a closed-form expression of the limiting temperature. However, the analytical results of this formalism do not seem to admit such a solution. This fact reveals the limitations of this continuum formalism. Often, consistent equations can be found between the original system and its continuum extrapolation, but it sometimes proves analytically intractable to produce results that exactly match those of the original system.

6. Discussion

Transitions from the discrete to the continuous abound in physics and typically result in simplified, more holistic dynamics for a system. A series of Hookean oscillators becomes a string governed by the wave equation [Pai05]. Coupled torsional pendula lead to the sine-Gordon equation [Sco69, DP06]. The master equation governs probabilities for discrete events, but when the events are made continuous, the Fokker Planck equation [VK92] becomes the equation of interest.

In each of these examples, the continuum limit transforms a system of many equations into a single partial differential equation. Inspired by these transformations, we showed that taking

[4] Alternatively, we might think that both numerator and denominator can be negative quantities, but for consistency with equation (38) we also want equation (87) to yield $\ln\bar{\eta} \simeq \ln\eta - \beta\lambda_1$ in the $\lambda_2 \to 0$ limit and such a limit requires $\ln\eta - \beta\lambda_1$ to be positive.

a class of statistical physics systems to the continuum limit led to a partial differential equation for the continuum analogues of the original partition functions.

The main result of this work is

$$Z_N(\beta) = \sum_{\boldsymbol{\sigma}} \omega\left(\sum\nolimits_i \sigma_i\right) \exp\left[-\beta \mathcal{E}\left(\sum\nolimits_i \sigma_i\right)\right]$$
$$\xrightarrow{\text{Continuum Limit}} \quad \left[\frac{\partial}{\partial \beta} + \mathcal{E}\left(\frac{\partial}{\partial \ln \eta}\right)\right] \mathcal{Z}(\eta;\beta) = 0 \tag{93}$$

where $\boldsymbol{\sigma}$ has components $\sigma_i \in \{0,1\}$. Unlike in the typical application of continuum limits where the system's degrees of freedom are transformed, here the discretely-valued partition function itself transitions to a continuous domain and we obtain a new way to study the thermal properties of a system: Instead of computing and analyzing the discrete sum definitive of the partition function, we solve the partial differential equation that governs the system's continuum extrapolation.

Importantly, the formalism applies only to mean-field systems where degrees of freedom are binary-valued (e.g. $\sigma_i \in \{0,1\}$). The foundational transformation equation (93) relies on the energy being a polynomial function of $\sum_i \sigma_i$, a quantity proportional to the mean field of the system. The fact that this analysis is limited in this way leads to an obvious question: Given that mean-field systems are typically already approximations of some original system, what value is there in pursuing a continuum-limit extrapolation (i.e. approximation) of an already approximated system? This question of value appears again when we consider that the results we derive from the continuum analogue are not new and simply affirm what we find in the original system. Why go through the work of making such a continuum extrapolation if it gives us the same thermal properties as the starting system, but often requires us to do more work to obtain these properties?

The value stems from that generally found in reformulations of known systems: It provides new ways to interpret the original system, which can serve as the foundation for connecting it to other systems that are far removed from the first system's context. As an explicit example, in section 5, we found that the continuum limit of mean-field systems with quadratic energies yielded a heat equation where inverse-temperature was the analogue for time. Thus, the temperature dependence of these systems could be defined as diffusion in degree-of-freedom space with all the concomitant interpretations that arise for heat evolution

$$\begin{array}{c}\text{Partition function of}\\ \text{interacting mean-field system}\end{array} \xrightarrow{\text{Continuum limit}} \begin{array}{c}\text{Temperature-dependent}\\ \text{diffusion in degree-of-freedom space.}\end{array} \tag{94}$$

For example, when the independent variable (t in the standard diffusion case and β in this case) is zero, the system is at 'maximum density.' For the partition function equation, this maximum density is reflected in the fact that all microstates are equally likely, and the partition function reduces to a sum over all microstates. However, as the independent variable increases (i.e. time evolves forward for standard diffusion or temperature is lowered for diffusion in degree-of-freedom space), the system density decreases, which for the partition function is a reflection of the fact that some higher energy microstates become less accessible at lower temperature. Thus, the transformation equation (93) allows us to relate systems that otherwise might seem independent, and these relations motivate new questions and interpretations about the original system.

This transformation also applies to more traditional mean feal models defined by ± 1 spin states. For example, in the traditional interacting mean-field Ising model [Yeo92] with energy

$-J\sum_{i,j} s_i s_j$ the spin sites $s_i \in \{-1,+1\}$ can be written as $s_i = 2\sigma_i - 1$, thus allowing the energy to be written in terms of $\sum_i \sigma_i$. However, the classic nearest-neighbor Ising model cannot be similarly translated.

Although we could reproduce the qualitative scaling behavior of the transition temperatures, we did not reproduce the functional dependencies of the order parameters. In other words, for all our continuum extrapolations, the analogs to thermal averages did not have the same functional dependence on the parameters as that in the original thermal system. Thus, it appears that there is a physical difference introduced into the partition function by taking it to the continuum limit. However, the phase transition of the original system is a sufficiently stark thermodynamic property that the transition's properties carry over. This suggests that on either side of this particular continuum limit are two different thermal models united by being part of the same universality class.

For the class of partition functions considered here, we required the energy function to be a finite power series of its argument. This limitation was to ensure that both the original power series and its continuum extrapolation remained finite. However, the "moment-to-partial derivative" transformation equation (14) is valid for any integer k, and thus, we should be able to translate all the terms of an infinite (but convergent) power series in $\mathcal{E}(k)$ to their continuum limit versions. This means if the energy potential is of the theoretical (and non-physical) form $\mathcal{E}(k) = \mathcal{E}_0 \cos(k)$, then we should be able to write the continuum limit analog as

$$\left[\frac{\partial}{\partial\beta} + \mathcal{E}_0 \cos\left(\frac{\partial}{\partial \ln\eta}\right)\right] \mathcal{Z}(\eta;\beta) = 0 \tag{95}$$

where $\cos(\cdots)$ is defined by its Taylor series expression. However, the physical properties of a statistical physics system with such an energy function are unknown, not to mention a reliable mathematical approach to solving the equation equation (95). Thus, such an infinite series energy function is primarily a mathematical curiosity.

Data availability statement

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Appendix A. Derivation of moment identity equation (12)

We aim to derive the identity equation (12) from the recursive identity equation (11). Namely, we want a closed-form expression for an arbitrary moment $\langle j^k\rangle_N$ that does not refer to lower-order moments. We rewrite equation (11) here for convenience:

$$\langle j^k\rangle_N Z_N(\beta) = N\langle j^{k-1}\rangle_N Z_N(\beta) - N\langle j^{k-1}\rangle_{N-1} Z_{N-1}(\beta). \tag{96}$$

To start, we define a new function $g_k(N)$ as

$$g_k(N) \equiv \langle j^k\rangle_N Z_N. \tag{97}$$

Then equation (96) can be written as

$$g_k(N) = N g_{k-1}(N) - N g_{k-1}(N-1). \tag{98}$$

From equation (97), we have the boundary conditions

$$g_0(N) = Z_N, \qquad g_k(0) = \delta(k,0), \tag{99}$$

where the second condition can be understood from equation (10).

We will employ a generating function approach to solving equation (98). We define the generating function $G(t,N)$ as

$$G(t,N) \equiv \sum_{k=0}^{\infty} t^k g_k(N). \tag{100}$$

From equation (99), we see that $G(t,N)$ has the boundary conditions:

$$G(0,N) = g_0(N), \qquad G(t,0) = 1. \tag{101}$$

Multiplying equation (98) by t^{k-1} and summing k from $k=1$ to ∞, we obtain

$$\frac{1}{t}\left(G(t,N) - g_0(N)\right) = NG(t,N) - NG(t,N-1), \tag{102}$$

and solving for $G(t,N)$ gives us

$$G(t,N) = \frac{g_0(N)}{1-Nt} - \frac{Nt}{1-Nt} G(t,N-1). \tag{103}$$

Iterating this recursion and using $G(t,0) = 1$, we find

$$G(t,N) = \sum_{m=0}^{N} (-1)^m g_0(N-m)\, t^m \prod_{i=0}^{m-1}(N-i) \prod_{j=0}^{m} \frac{1}{(1-(N-j)t)}. \tag{104}$$

We can condense the first product within the summation by using the falling factorial definition $(N)_m \equiv \prod_{i=0}^{m-1}(N-i)$. For the other product, we introduce the complete homogeneous symmetric polynomial h_j defined explicitly as

$$h_j(X_1,X_2,\ldots,X_N) \equiv \sum_{\ell_i \geqslant 0} \delta(j,\ell_1+\ell_2+\cdots+\ell_N)\, X_1^{\ell_1} X_2^{\ell_2} \cdots X_N^{\ell_N}, \tag{105}$$

and implicitly as

$$\prod_{k=1}^{N} \frac{1}{1-X_k t} = \sum_{j=0}^{\infty} t^j h_j(X_1,X_2,\ldots,X_N). \tag{106}$$

Returning to equation (104), we can take the t^m factor and the second product to obtain

$$t^m \prod_{j=0}^{m} \frac{1}{1-(N-j)t} = \frac{t^m}{(1-Nt)^{m+1}} \prod_{j=0}^{m} \frac{1}{1+jt/(1-Nt)}$$
$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{t^{\ell+m}}{(1-Nt)^{\ell+m+1}} h_{\ell}(1,2,\ldots,m), \tag{107}$$

where in the final line we used equation (106) and $h_\ell(0,1,2,\ldots,m) = h_\ell(1,2,\ldots,m)$. Now, with the identity

$$\frac{t^{\ell+m}}{(1-Nt)^{\ell+m+1}} = \frac{1}{N^{\ell+m}} \sum_{k=\ell+m}^{\infty} \binom{k}{\ell+m} (Nt)^k, \tag{108}$$

we then find

$$t^m \prod_{j=0}^{m} \frac{1}{1-(N-j)t} = \sum_{\ell=0}^{\infty} (-1)^{\ell} N^{-(\ell+m)} \sum_{k=\ell+m}^{\infty} \binom{k}{\ell+m} (Nt)^k h_{\ell}(1,2,\ldots,m)$$
$$= \sum_{r=m}^{\infty} (-1)^{r-m} N^{-r} \sum_{k=r}^{\infty} \binom{k}{r} (Nt)^k h_{r-m}(1,2,\ldots,m), \tag{109}$$

where we changed summation variables from ℓ to $r-m$ in the final line. Substituting this reduced expression for the product into equation (104) gives us

$$G(t,N) = \sum_{m=0}^{N} (-1)^m g_0(N-m)(N)_m \sum_{r=m}^{\infty} (-1)^{r-m} N^{-r} \sum_{k=r}^{\infty} \binom{k}{r} (Nt)^k h_{r-m}(1,2,\ldots,m)$$
$$= \sum_{m=0}^{N} (-1)^m g_0(N-m)(N)_m \sum_{k=m}^{\infty} \binom{k}{r} (Nt)^k \sum_{r=m}^{k} (-1)^{r-m} N^{-r} h_{r-m}(1,2,\ldots,m)$$
$$= \sum_{k=0}^{\infty} t^k \sum_{m=0}^{k} g_0(N-m)(N)_m \sum_{r=m}^{k} (-1)^r N^{k-r} \binom{k}{r} h_{r-m}(1,2,\ldots,m). \tag{110}$$

In rearranging the order of the summations, we repeatedly used the identity $\sum_{j=A}^{M}\sum_{i=j}^{\infty} = \sum_{i=A}^{\infty}\sum_{j=A}^{i}$, where M is positive and related to the positive integer A through $A \leqslant M$. Isolating the coefficient of the t^k term in equation (110), we find

$$g_k(N) = \sum_{m=0}^{k} g_0(N-m)(N)_m \sum_{r=m}^{k} (-1)^r N^{k-r} \binom{k}{r} h_{r-m}(1,2,\ldots,m). \tag{111}$$

This result is close to our desired form for $g_k(N)$, but the polynomials h_j can be further simplified by writing them in terms of Stirling numbers of the second kind. To make this simplification, we use the identity [Wik23]

$$S(k+\ell,\ell) = h_k(1,2,\ldots,\ell), \tag{112}$$

where $S(n,m)$ is the Stirling number of the second kind. Using this, we obtain finally

$$g_k(N) = \sum_{m=0}^{k} g_0(N-m)(N)_m \sum_{r=m}^{k} (-1)^r N^{k-r} \binom{k}{r} S(r,m). \tag{113}$$

And writing this result in terms of partition functions via equation (97), we then have

$$\langle j^k \rangle_N Z_N = \sum_{m=0}^{k} Z_{N-m} (N)_m \sum_{r=m}^{k} (-1)^r N^{k-r} \binom{k}{r} S(r,m), \tag{114}$$

thus affirming equation (12).

Appendix B. Derivation of moment to partial derivative identity equation (14)

We seek to derive equation (14) (reproduced here)

$$\langle \left(\sum_i \sigma_i\right)^k \rangle_N Z_N(\beta) \longrightarrow \left(\frac{\partial}{\partial \ln \eta}\right)^k \mathcal{Z}(\eta;\beta) \tag{115}$$

by applying the continuum limit transformations

$$Z_{N-j}(\beta) \to \mathcal{Z}(\eta - ja;\beta) \text{ for } j \in \mathbb{Z} \quad \text{and} \quad N \to \eta/a \tag{116}$$

to equation (12) and then taking the limit of the result as $a \to 0$.

Before we apply the continuum limit transformations to equation (12), we will prepare for the future application of an identity by expressing $(N)_\ell$ in terms of Stirling numbers of the first kind $s(n,m)$ [Weib]:

$$(N)_\ell = \prod_{i=0}^{\ell-1} (N-i) = \sum_{j=0}^{\ell} s(\ell,j) N^j. \tag{117}$$

Equation (12) then becomes

$$\langle j^k \rangle_N Z_N = \sum_{\ell=0}^{k} Z_{N-\ell} \sum_{j=0}^{\ell} s(\ell,j) N^j \sum_{r=\ell}^{k} (-1)^r N^{k-r} \binom{k}{r} S(r,\ell). \tag{118}$$

Making the continuum limit transformations gives us

$$\langle j^k \rangle_N Z_N \to \lim_{a\to 0} \sum_{\ell=0}^{k} \mathcal{Z}(\eta - \ell a) \sum_{j=0}^{\ell} s(\ell,j) (\eta/a)^j \sum_{r=\ell}^{k} (-1)^r (\eta/a)^{k-r} \binom{k}{r} S(r,\ell). \tag{119}$$

From here, we note that by the limit definition of higher order derivatives, we know

$$\frac{\partial^k}{\partial \eta^k} \mathcal{Z}(\eta) = \lim_{a\to 0} a^{-k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \mathcal{Z}(\eta - \ell a). \tag{120}$$

Inverting this equation, we find that in order for equation (120) to be valid, $\mathcal{Z}(\eta - \ell a)$ must satisfy

$$\mathcal{Z}(\eta - \ell a) = \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} a^m \frac{\partial^m}{\partial \eta^m} \mathcal{Z}(\eta). \tag{121}$$

Substituting equation (121) into equation (119), we find

$$\begin{aligned}\langle j^k \rangle_N Z_N \to \lim_{a\to 0} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} (-1)^m \binom{\ell}{m} a^m \frac{\partial^m}{\partial \eta^m} \mathcal{Z}(\eta) \\ \times \sum_{j=0}^{\ell} s(\ell,j) (\eta/a)^j \sum_{r=\ell}^{k} (-1)^r (\eta/a)^{k-r} \binom{k}{r} S(r,\ell).\end{aligned} \tag{122}$$

Switching the order of the first two summations and collecting factors of a yield

$$\begin{aligned}\langle j^k \rangle_N Z_N \to \lim_{a\to 0} \sum_{m=0}^{k} \sum_{\ell=m}^{k} (-1)^m \binom{\ell}{m} \frac{\partial^m}{\partial \eta^m} \mathcal{Z}(\eta) \\ \times \sum_{j=0}^{\ell} s(\ell,j) \eta^{j+k-r} \sum_{r=\ell}^{k} (-1)^r a^{m-j-k+r} \binom{k}{r} S(r,\ell).\end{aligned} \tag{123}$$

By the limit definition

$$\lim_{a\to 0} a^N = \delta(N,0), \tag{124}$$

we find

$$\begin{aligned}\langle j^k \rangle_N Z_N &\to \sum_{m=0}^{k} \sum_{\ell=m}^{k} (-1)^m \binom{\ell}{m} \frac{\partial^m}{\partial \eta^m} \mathcal{Z}(\eta) \sum_{j=0}^{\ell} s(\ell,j) \eta^{j+k-r} \sum_{r=\ell}^{k} (-1)^r \delta(r, k+j-m) \binom{k}{r} S(r,\ell) \\ &= \sum_{m=0}^{k} \eta^m \frac{\partial^m}{\partial \eta^m} \mathcal{Z}(\eta) \sum_{\ell=m}^{k} \binom{\ell}{m} \sum_{j=0}^{\ell} s(\ell,j) (-1)^{k+j} \binom{k}{k+j-m} S(k+j-m,\ell) \\ &= \sum_{m=0}^{k} \eta^m \frac{\partial^m}{\partial \eta^m} \mathcal{Z}(\eta)\, \Omega_{km},\end{aligned} \tag{125}$$

where we defined

$$\Omega_{km} = \sum_{\ell=m}^{k} \binom{\ell}{m} \sum_{j=0}^{\ell} (-1)^{k+j} \binom{k}{k+j-m} s(\ell,j) S(k+j-m,\ell). \tag{126}$$

From here, we will use the Egorychev method of proving combinatorial identities by representing quantities as contour integrals [Ego84]. The subsequent derivation follows that given in [Rei23].

The contour integral expressions for the Stirling numbers of the first and second kind ([Gou], equations (7.1) and (7.3), respectively) are

$$S(M,\ell) = \frac{M!}{2\pi \mathrm{i}\, \ell!} \oint_\Gamma \frac{\mathrm{d}z}{z^{M+1}} (\mathrm{e}^z - 1)^\ell, \qquad s(\ell,j) = \frac{\ell!}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathrm{d}q}{q^{j+1}} \binom{q}{\ell}. \tag{127}$$

where Γ is a closed contour encircling the origin in the complex plane. With these identities equation (126) becomes

$$\Omega_{km} = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z}\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}q}{q}\sum_{\ell=m}^{k}\binom{\ell}{m}\sum_{j=0}^{\ell}(-1)^{k+j}\frac{k!}{(m-j)!}\frac{1}{q^j}\binom{q}{\ell}\frac{1}{z^{k+j-m}}(e^z-1)^\ell$$
$$= \frac{k!}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{k+1}}\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}q}{q^{m+1}}\sum_{\ell=m}^{k}\binom{q}{\ell}\binom{\ell}{m}(e^z-1)^\ell(-1)^{k+m}\sum_{j=0}^{\ell}(-1)^{j-m}\frac{1}{(m-j)!}\frac{1}{q^{j-m}}\frac{1}{z^{j-m}}. \tag{128}$$

Isolating the sum over j yields

$$\sum_{j=0}^{\ell}(-1)^{j-m}\frac{1}{(m-j)!}\frac{1}{q^{j-m}}\frac{1}{z^{j-m}} \to \sum_{j=-\infty}^{m}\frac{1}{(m-j)!}(-qz)^{m-j} = \mathrm{e}^{-qz}. \tag{129}$$

In equation (129), we extended the lower limit of the summation to $-\infty$. This extension does not change the value of Ω_{km} since $s(\ell,j)=0$ for $j<0$ as can be affirmed by the contour integral expression equation (127). We also stopped the upper limit at $j=m$ since equation (126) implies that j cannot be greater than m. Thus equation (128) becomes

$$\Omega_{km} = \frac{k!}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{k+1}}\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}q}{q^{m+1}}\sum_{\ell=m}^{k}\binom{q}{\ell}\binom{\ell}{m}(\mathrm{e}^z-1)^\ell(-1)^{k+m}\mathrm{e}^{-qz}. \tag{130}$$

For the summation over ℓ, the standard binomial identity

$$\sum_{\ell=m}^{k}\binom{q}{\ell}\binom{\ell}{m}X^\ell = \binom{q}{m}X^m(1+X)^{q-m} \tag{131}$$

leads to

$$\sum_{\ell=m}^{k}\binom{q}{\ell}\binom{\ell}{m}(e^z-1)^\ell = \binom{q}{m}(\mathrm{e}^z-1)^m\mathrm{e}^{zq-zm}. \tag{132}$$

Equation (130) is then

$$\Omega_{km} = \frac{k!}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{k+1}}\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}q}{q^{m+1}}\binom{q}{m}(\mathrm{e}^z-1)^m(-1)^{k+m}\mathrm{e}^{zq-zm}\mathrm{e}^{-qz}$$
$$= \frac{k!}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z}{z^{k+1}}(\mathrm{e}^{-z}-1)^m(-1)^k\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}q}{q^{m+1}}\binom{q}{m}$$
$$= \frac{k!}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}z'}{z'^{k+1}}\left(\mathrm{e}^{z'}-1\right)^m\cdot\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}q}{q^{m+1}}\binom{q}{m}$$
$$= S(k,m)\cdot s(m,m) = S(k,m). \tag{133}$$

In the penultimate line, we defined $z'=-z$. Now returning to equation (125), we have

$$\langle j^k\rangle_N Z_N \to \sum_{m=0}^{k}\eta^m\frac{\partial^m}{\partial\eta^m}\mathcal{Z}(\eta)\,S(k,m). \tag{134}$$

Using the recursive definition of the Stirling polynomial ($S(k+1,m)=S(k,m)m+S(k,m-1)$) one can employ a proof by induction (see appendix C) to show

$$\frac{\partial^\ell}{\partial\ln\eta^\ell}F(\eta) = \sum_{m=0}^{\ell}S(\ell,m)\,\eta^m\frac{\partial^m}{\partial\eta^m}F(\eta), \tag{135}$$

for a function $F(\eta)$. This result is shown less directly in [Boy12]. Thus, in all, we see that taking equation (134) to the continuum limit yields the transformation

$$\langle j^k\rangle_N Z_N \to \left(\frac{\partial}{\partial \ln \eta}\right)^k \mathcal{Z}(\eta)\,. \tag{136}$$

Appendix C. Logarithmic derivatives and Grunert's formula

Here, we prove, by induction, equation (135), the expression of higher-order logarithmic derivatives as a linear combination of dimensionless derivative operators:

$$\frac{\partial^\ell}{\partial \ln \eta^\ell} F(\eta) = \sum_{m=0}^{\ell} S(\ell, m)\, \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta) \tag{137}$$

where $S(\ell,k)$ are Stirling numbers of the second kind, and $F(\eta)$ is a real-valued function.

As the first inductive step, we show that the result is true for $\ell = 1$. Doing so yields,

$$\frac{\partial}{\partial \ln \eta} F(\eta) = S(1,0)\, F(\eta) + S(1,1)\, \eta \frac{\partial}{\partial \eta} F(\eta) = \eta \frac{\partial}{\partial \eta} F(\eta)\,, \tag{138}$$

where we used $S(\ell,0) = \delta_{\ell 0}$ and $S(1,1) = 1$ [Weic]. Assuming the identity is true for $\ell = k$, we have

$$\frac{\partial^k}{\partial \ln \eta^k} F(\eta) = \sum_{m=0}^{k} S(k, m)\, \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta)\,. \tag{139}$$

Now, to prove the result is true for $\ell = k+1$. Taking an additional logarithmic derivative on both sides of equation (139), we obtain

$$\begin{aligned}
\frac{\partial^{k+1}}{\partial \ln \eta^{k+1}} F(\eta) &= \sum_{m=0}^{k} S(k,m)\, \eta \frac{\partial}{\partial \eta}\left(\eta^m \frac{\partial}{\partial \eta^m} F(\eta)\right) \\
&= \sum_{m=0}^{k} S(k,m)\, m\, \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta) + \sum_{m=0}^{k} S(k,m)\, \eta^{m+1} \frac{\partial^{m+1}}{\partial \eta^{m+1}} F(\eta) \\
&= \sum_{m=1}^{k} \left(S(k,m)\, m + S(k,m-1)\right) \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta) + S(k,k)\, \eta^{k+1} \frac{\partial^{k+1}}{\partial \eta^{k+1}} F(\eta) \\
[\text{Stirling Number Identity}] &= \sum_{m=1}^{k} S(k+1,m)\, \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta) + S(k,k)\, \eta^{k+1} \frac{\partial^{k+1}}{\partial \eta^{k+1}} F(\eta) \\
[\text{Using } S(n,0) = \delta_{n0}] &= \sum_{m=0}^{k} S(k+1,m)\, \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta) + S(k,k)\, \eta^{k+1} \frac{\partial^{k+1}}{\partial \eta^{k+1}} F(\eta) \\
[\text{Using } S(k,k) = 1] &= \sum_{m=0}^{k+1} S(k+1,m) \eta^m \frac{\partial^m}{\partial \eta^m} F(\eta),
\end{aligned} \tag{140}$$

which is the desired result for the final induction step.

Appendix D. Alternative derivation of equation (46)

We originally obtained equation (46) from a straightforward application of Green's functions, but it is possible to derive the result by employing the Hubbard-Stratonovich method [Hub59]. In what follows, we will use the superscripts (1) and (2) to distinguish partition functions obtained from linear and quadratic energy functions, respectively.

For the condition $\lambda_2 = -|\lambda_2|$, the energy function is $\mathcal{E}(j) = \lambda_1 j - |\lambda_2| j^2/2$ and the associated partition function is

$$Z_N^{(2)}(\beta\lambda_1, \beta|\lambda_2|) = \sum_{j=0}^{N} \binom{N}{j} \omega(j)\, \mathrm{e}^{-\beta\mathcal{E}_N(j)} = \sum_{j=0}^{N} \binom{N}{j} \omega(j)\, \mathrm{e}^{-\beta\lambda_1 j + \beta|\lambda_2| j^2/2}. \tag{141}$$

With a Gaussian identity, we then find

$$\begin{aligned} Z_N^{(2)}(\beta\lambda_1, \beta|\lambda_2|) &= \frac{1}{\sqrt{2\pi|\lambda_2|\beta}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/4\beta|\lambda_2|} \sum_{j=0}^{N} \binom{N}{j} \omega(j)\, \mathrm{e}^{-j(\beta\lambda_1 + x)} \\ &= \frac{1}{\sqrt{4\pi|\beta\lambda_2|}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2\beta|\lambda_2|} Z_N^{(1)}(\beta\lambda_1 + x), \end{aligned} \tag{142}$$

where we defined $Z_N^{(1)}(x) \equiv \sum_{j=0}^{N} \binom{N}{j} \omega(j) \mathrm{e}^{-jx}$. Next, we note that from equation (3), the partition function $Z^{(1)}(x)$ satisfies

$$\partial_x Z_N^{(1)}(x) = -N\left(Z_N^{(1)}(x) - Z_{N-1}^{(1)}(x)\right), \tag{143}$$

which, when extrapolated to the continuum regime, yields the differential equation

$$\left(\frac{\partial}{\partial\beta} + \lambda_1 \frac{\partial}{\partial \ln\eta}\right) \mathcal{Z}^{(1)}(\eta; \beta\lambda_1) = 0, \tag{144}$$

with $\mathcal{Z}^{(1)}(\eta; 0) = \Omega(\eta)$. Any continuously differentiable function f of the variable $\ln\eta - \beta\lambda_1$ satisfies the differential equation (144) (e.g. $\mathcal{Z}^{(1)}(\eta, \beta\lambda_1) = \sinh(\ln\eta - \beta\lambda_1)$ is a solution), but to find the *particular* solution we must use our boundary condition $\mathcal{Z}^{(1)}(\eta; 0)$. Doing so, we find the particular solution to equation (144) is

$$\mathcal{Z}^{(1)}(\eta; \beta\lambda_1) = \Omega\left(\eta \mathrm{e}^{-\beta\lambda_1} + 1\right). \tag{145}$$

We complete the continuous extrapolation of $Z_N^{(2)}(\beta\lambda_1, \beta\lambda_2)$ to $\mathcal{Z}^{(2)}(\eta; \beta\lambda_1, \beta\lambda_2)$ in equation (142) by making the two transformations

$$Z_N^{(1)}(\beta\lambda_1 + x) \to \mathcal{Z}^{(1)}(\eta; \beta\lambda_1 + x) \tag{146}$$

$$Z_N^{(2)}(\beta\lambda_1, \beta\lambda_2) \to \mathcal{Z}^{(2)}(\eta; \beta\lambda_1, \beta\lambda_2). \tag{147}$$

With equation (145), and the above transformations, the continuum extrapolation of equation (142) then becomes

$$\begin{aligned} \mathcal{Z}^{(2)}(\eta; \beta\lambda_1, \beta\lambda_2) &= \frac{1}{\sqrt{2\pi|\lambda_2|\beta}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2\beta|\lambda_2|}\, \Omega\left(\eta \mathrm{e}^{-\beta\lambda_1 - x} + 1\right) \\ &= \frac{1}{\sqrt{2\pi|\lambda_2|\beta}} \int_{-\infty}^{\infty} \mathrm{d}\ln\eta' \exp\left[-\frac{1}{2\beta|\lambda_2|} \ln^2\left(\frac{\eta \mathrm{e}^{-\beta\lambda_1}}{\eta'}\right)\right] \Omega(\eta' + 1) \end{aligned} \tag{148}$$

where in the final equality we changed variables using $x = \ln(\eta e^{-\beta\lambda_1}/\eta')$. This result thus reproduces equation (46).

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